Calabi-Yau-threefolds with Picard number $\rho(X) = 2$

and their Kähler cone I

Marco Kühnel

Fachbereich Mathematik, Universität Bayreuth, 95440 Bayreuth, Germany

Received: 9 October 2001; in final form: 27 November 2002 /
Published online: 1 July 2003 – © Springer-Verlag 2003

Abstract. We describe properties of the Kähler cone of general Calabi-Yau-threefolds with Picard number $\rho(X) = 2$ and prove the rationality of the Kähler cone, if $X$ is a Calabi-Yau-hypersurface in a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$ and $c_3(X) \leq -54$. Without the latter assumption we prove the positivity of $c_2(X)$.

1 Introduction

In this paper, a Calabi-Yau-threefold is a compact complex Kähler manifold of dimension three with $K_X = O_X$ and $H^1(O_X) = 0$.

Wilson stated in 1994 [Wi94b] the conjecture that the Kähler cone of a Calabi-Yau-threefold $X$ is rational and finitely generated in $N_1(X)$, if $c_2(X)$ is positive, i.e. $D.c_2(X) > 0$ for every nef divisor $D$.

In the present paper we deal with the case of Picard number $\rho(X) = 2$. If $X \rightarrow S$ is an elliptic fibration onto a smooth surface and $\rho(X) = 2$, then $S \cong \mathbb{P}^2$. Oguiso proves in [Og93] that for every elliptic fibration $X \rightarrow S$, the surface $S$, smooth or not, is rational. Moreover, it is an easy argument to verify that $C = \mathbb{P}^1$, if $X \rightarrow C$ is a fibration onto a normal curve $C$ and $X$ is a Calabi-Yau manifold. So it is natural to consider projective spaces as base spaces of Calabi-Yau-fibrations.

A rather complete picture about the Kähler cone should be obtained for those $X$ which can be embedded in $\mathbb{P}^n$-bundles over $\mathbb{P}^m$ as hypersurfaces. Here we are considering the case $(n, m) = (2, 2)$, i.e. $X$ is a family of cubic elliptic curves. We will prove the rationality of the Kähler cone for most cases:

Result 2 Let $\mathcal{E} \rightarrow \mathbb{P}^2$ be a rank-3-bundle, $Z := \mathbb{P}(\mathcal{E})$, $X \subset |-K_Z|$ a Calabi-Yau threefold with $\rho(X) = 2$ and $\kappa(Z, -K_Z) > 0$. Then $\partial K(X)$ is rational.

* The author acknowledges gratefully support by the DFG priority program ‘Global Methods in Complex Geometry’.
There is also a topological criterion on $X$ for the condition $\kappa(Z, -K_Z) > 0$, which in particular includes all possible $Z$ with $-K_Z$ nef (cf. Remark 4.15):

**Result 3** Let $\mathcal{E} \rightarrow \mathbb{P}^2$ be a rank-3-bundle, $X \subset \mathbb{P}(\mathcal{E}) := Z$ a Calabi-Yau threefold with $\rho(X) = 2$. If $c_3(X) \leq -54$ or $\mathcal{E}$ is not simple, then $h^0(-K_Z) > 1$, in particular $\partial K(X)$ is rational.

In order to prove the rationality of $\partial K(X)$, we investigate, whether

$$K(X) = K(Z)|X$$

holds, if $Z$ denotes the $\mathbb{P}^n$-bundle. A result of Kollár states that this holds, if $Z$ is Fano. We generalize this assumption and describe the situation $K(X) \neq K(Z)|X$.

**Result 1** Let $\mathcal{E} \rightarrow \mathbb{P}^2$ be a rank-3-bundle, $X \subset \mathbb{P}(\mathcal{E}) := Z$ a Calabi-Yau-3-fold and $\rho(X) = 2$. Then

$$K(X) = K(Z)|X$$

holds, unless:

$-K_Z$ is big and nef, but not ample, $-K_Z|X$ is ample and there exists a surface $G \subset Z$ such that $X \cap G = \emptyset$ and

$$[\mu G] = 9\mathcal{O}_Z(1)^2 - (6c_1(\mathcal{E}).h + 9)\mathcal{O}_Z(1).p^*h$$
$$+ (9c_2(\mathcal{E}) + 3c_1(\mathcal{E}).h + 9 - 2c_1^2(\mathcal{E})) F$$

for a certain $\mu > 0$.

This part uses the bundle-situation only for the details of the description. Furthermore, we show that $\partial K(Z)$ is rational, if $\kappa(Z, -K_Z) > 0$. Hence, under this assumption, it is enough to discuss the cases $K(X) \neq K(Z)|X$ in order to prove rationality of $\partial K(X)$. This is done by using some vector bundle theory, hence is essentially connected to the embedding into a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$.

In the last part, we prove

**Result 4** Let $\mathcal{E} \rightarrow \mathbb{P}^2$ be a rank-3-bundle, $X \subset \mathbb{P}(\mathcal{E})$ a Calabi-Yau threefold and $\rho(X) = 2$. Then

$$D.c_2(X) > 0 \text{ for all } D \in K(X)$$

Thus we are confirming the Wilson conjecture for hypersurfaces in $\mathbb{P}^2$-bundles over $\mathbb{P}^2$ with $c_3(X) \leq -54$ and $\rho(X) = 2$. Also for the positivity of $c_2(X)$ we use the bundle information strongly.

The cases $(n, m) = (3, 1)$ and $(n, m) = (1, 3)$ are dealt with in [Kü02]. Moreover, the case of a family of quartic elliptic curves also may be interesting.

Calabi-Yau manifolds with $\rho(X) = 2$ are also considered in Mirror Symmetry as mirrors of two-parameter-families of Calabi-Yau manifolds, which are constructed in order to choose a good one-parameter family.

This article grew out of the author’s thesis at the University of Bayreuth. I would like to express my gratitude to Prof. Thomas Peternell for all his support and valuable advice.