Conformal Killing forms on Riemannian manifolds

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Abstract. Conformal Killing forms are a natural generalization of conformal vector fields on Riemannian manifolds. They are defined as sections in the kernel of a conformally invariant first order differential operator. We show the existence of conformal Killing forms on nearly Kähler and weak $G_2$-manifolds. Moreover, we give a complete description of special conformal Killing forms. A further result is a sharp upper bound on the dimension of the space of conformal Killing forms.

1 Introduction

A classical object of differential geometry are Killing vector fields. These are by definition infinitesimal isometries, i.e. the flow of such a vector field preserves a given metric. The space of all Killing vector fields forms the Lie algebra of the isometry group of a Riemannian manifold and the number of linearly independent Killing vector fields measures the degree of symmetry of the manifold. It is known that this number is bounded from above by the dimension of the isometry group of the standard sphere and, on compact manifolds, equality is attained if and only if the manifold is isometric to the standard sphere or the real projective space. Slightly more generally one can consider conformal vector fields, i.e. vector fields with a flow preserving a given conformal class of metrics. There are several geometric conditions which force a conformal vector field to be Killing.

Much less is known about a rather natural generalization of conformal vector fields, the so-called conformal Killing forms. These are $p$-forms $\psi$ satisfying for any vector field $X$ the differential equation

[Equation]

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\[
\n\nabla_X \psi - \frac{1}{p+1} X \lrcorner \, d\psi + \frac{1}{n-p+1} X^* \wedge d^* \psi = 0, \quad (1.1)
\]

where \( n \) is the dimension of the manifold, \( \nabla \) denotes the covariant derivative of the Levi-Civita connection, \( X^* \) is 1-form dual to \( X \) and \( \lrcorner \) is the operation dual to the wedge product. It is easy to see that a conformal Killing 1-form is dual to a conformal vector field. Coclosed conformal Killing \( p \)-forms are called \textit{Killing forms}. For \( p = 1 \) they are dual to Killing vector fields.

The left hand side of equation (1.1) defines a first order elliptic differential operator \( T \), which was already studied in the context of Stein-Weiss operators (c.f. [6]). Equivalently one can describe a conformal Killing form as a form in the kernel of \( T \). From this point of view conformal Killing forms are similar to twistor spinors in spin geometry. One shared property is the conformal invariance of the defining equation. In particular, any form which is parallel for some metric \( g \), and thus a Killing form for trivial reasons, induces non-parallel conformal Killing forms for metrics conformally equivalent to \( g \) (by a non-trivial change of the metric).

Killing forms, as a generalization of the Killing vector fields, were introduced by K. Yano in [30], Later S. Tachibana (c.f. [25]), for the case of 2-forms, and more generally T. Kashiwada (c.f. [17], [18]) introduced conformal Killing forms generalizing conformal vector fields.

Already K. Yano noted that a \( p \)-form \( \psi \) is a Killing form if and only if for any geodesic \( \gamma \) the \((p-1)\)-form \( \dot{\gamma} \lrcorner \psi \) is parallel along \( \gamma \). In particular, Killing forms give rise to quadratic first integrals of the geodesic equation, i.e. functions which are constant along geodesics. Hence, they can be used to integrate the equation of motion. This was first done in the article [22] of R. Penrose and M. Walker, which initiated an intense study of Killing forms in the physics literature. In particular, there is a local classification of Lorentz manifolds with Killing 2-forms. More recently Killing forms and conformal Killing forms have been successfully applied to define symmetries of field equations (c.f. [3], [4]).

Despite this longstanding interest in (conformal) Killing forms there are only very few global results on Riemannian manifolds. Moreover the number of the known non-trivial examples on compact manifolds is surprisingly small. The aim of this article is to fill this gap and to start a study of global properties of conformal Killing forms.

As a first contribution we will show that there are several classes of Riemannian manifolds admitting Killing forms, which so far did not appear in the literature. In particular, we will show that there are Killing forms on nearly Kähler manifolds and on manifolds with a weak \( G_2 \)-structure. All these examples are related to Killing spinors and nearly parallel vector cross products. Moreover, they are all so-called \textit{special Killing forms}. The restriction from Killing forms to special Killing forms is analogous to the definition of a Sasakian structure as a unit length Killing vector field satisfying an additional equation. One of our main results in this paper is the complete description of manifolds admitting special Killing forms.

Since conformal Killing forms are sections in the kernel of an elliptic operator it is clear that they span a finite dimensional space in the case of compact manifolds. Our second main result is an explicit upper bound for the dimension of the space of conformal Killing forms on arbitrary connected Riemannian manifolds. The upper