Continuation of $L^2$-holomorphic functions

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Abstract Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and let $L^2_h(\Omega)$ be the $L^2$-holomorphic functions on $\Omega$. We show that the envelope of holomorphy and the $L^2_h(\Omega)$-envelope of holomorphy of $\Omega$ differ by at most a pluripolar set.

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1 Introduction

Let $\Omega$ be a domain in $\mathbb{C}^n$. We denote the holomorphic functions on $\Omega$ by $\mathcal{O}(\Omega)$ and set $L^2_h(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$.

The goal of this work is to study analytic continuation of functions in $L^2_h(\Omega)$. Pflug [P] showed that if $\Omega$ is a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\Omega$ is fat (i.e. $\text{int} \bar{\Omega} = \Omega$), then $\Omega$ is an $L^2_h(\Omega)$-domain of holomorphy. On the other hand, if $E$ is an analytic subset of a domain $\Omega$, then Bell [B] showed that all functions in $L^2_h(\Omega)$ extend holomorphically across $E$. In one variable, a complete description of these phenomena has been known for a while. With the terminology from the next section, it can be stated as follows.

Theorem 1. Let $\Omega$ be a domain in $\mathbb{C}$. Then $\Omega$ is an $L^2_h$-domain of holomorphy if and only if $\Omega$ does not have any polar boundary points.

See [C] for a proof. We remark that this is a local description. In Theorem 7 we prove a generalization of this result to Riemann domains over $\mathbb{C}^n$ with bounded projection. Using this we get the following description of $L^2_h$-envelopes of holomorphy.

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Theorem 2 (Main theorem). Let \((X, \pi)\) be a Riemann domain over \(\mathbb{C}^n\) such that \(\pi(X)\) is bounded. Then the envelope of holomorphy, \((\tilde{X}, \tilde{\pi})\), can be embedded into the \(L^2_h(X)\)-envelope of holomorphy, \((\tilde{X}_{L^2_h}, \tilde{\pi}_{L^2_h})\), and the difference between the two sets is at most pluripolar.

The article is organized as follows. We begin with a section covering the necessary definitions and background results. Section 3 proves a version of Theorem 1 for schlicht bounded domains. The technicalities of the Riemann domain case are treated in Section 4. We discuss the main theorem in the last section.

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2 Definitions and background results

Let \(\Omega\) be a domain in \(\mathbb{C}^n\). We say that \(\Omega\) is an \(L^2_h(\Omega)\)-domain of holomorphy, or an \(L^2_h\)-domain for short, if there are functions in \(L^2_h(\Omega)\) which do not extend holomorphically. More precisely:

**Definition 1.** Let \(\Omega\) be a domain in \(\mathbb{C}^n\). We will call \(\Omega\) an \(L^2_h\)-domain of holomorphy if we cannot find non-empty open sets \(U\) and \(V, V\) connected and not contained in \(\Omega\) and \(U \subset V \cap \Omega\), such that for every \(f \in L^2_h(\Omega)\) there is a function \(F \in \mathcal{O}(V)\) satisfying \(F = f\) on \(U\).

Let us remark that we require \(F\) to be only holomorphic on \(V\), not \(L^2\).

If \(\Omega\) is not an \(L^2_h\)-domain, then all functions in \(L^2_h(\Omega)\) extend holomorphically. These extensions need not, however, be single valued, so we introduce Riemann domains, i.e. domains over \(\mathbb{C}^n\) with a local homeomorphism to \(\mathbb{C}^n\) called the projection. Let us recall that two Riemann domains are equivalent if there is a biholomorphism between them which preserves the projections. We say that a Riemann domain \((X, \pi)\) is an \(L^2_h\)-domain if any other Riemann domain to which all the functions in \(L^2_h(X)\) extend holomorphically is equivalent to \((X, \pi)\). For domains in \(\mathbb{C}^n\), this is equivalent to the definition given above. A Riemann domain which is equivalent to a domain in \(\mathbb{C}^n\) is said to be schlicht.

If \((X, \pi)\) is not an \(L^2_h\)-domain, then by Thullen’s theorem (see for example [N]) there exists a unique largest Riemann domain to which all functions in \(L^2_h(X)\) extend holomorphically. We call this the \(L^2_h(X)\)-envelope of holomorphy of \((X, \pi)\), or \(L^2_h\)-envelope for short, and denote it \((\tilde{X}_{L^2_h}, \tilde{\pi}_{L^2_h})\).

Recall that a set is (pluri)polar if there is a (pluri)subharmonic function \(u\), not identically \(-\infty\), such that \(E \subset \{u = -\infty\}\). When we talk about polar sets in \(\mathbb{C}^n\), we will mean polar in the sense of \(\mathbb{R}^{2n}\). In order to describe the boundary of \(L^2_h\)-domains we introduce the following definition.

**Definition 2.** Let \(\zeta\) be a boundary point of a domain \(\Omega \in \mathbb{C}^n\). We will say that \(\zeta\) is a pluripolar boundary point if there is a neighborhood \(V\) of \(\zeta\) such that \(\partial \Omega \cap V\) is a pluripolar set.

For Riemann domains, let us first recall Grauert’s [G] notion of boundary points.