Weighted Strichartz estimates for the wave equation in even space dimensions

Jun Kato\textsuperscript{1,\*}, Tohru Ozawa\textsuperscript{2}

\textsuperscript{1} Mathematical Institute, Tohoku University, Sendai 980-8578, Japan
\textsuperscript{2} Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

Received: 13 November 2002; in final form: 27 September 2003 / Published online: 17 February 2004 – © Springer-Verlag 2004

Dedicated to Professor Mitsuru Ikawa on the occasion of his sixtieth birthday

Abstract We prove the weighted Strichartz estimates for the wave equation in even space dimensions with radial symmetry in space. Although the odd space dimensional cases have been treated in our previous paper [5], the lack of the Huygens principle prevents us from a similar treatment in even space dimensions. The proof is based on the two explicit representations of solutions due to Rammaha [11] and Takamura [14] and to Kubo-Kubota [6]. As in the odd space dimensional cases [5], we are also able to construct self-similar solutions to semilinear wave equations on the basis of the weighted Strichartz estimates.

Mathematics Subject Classification (2000): 35L05, 35B45, 35L70

1 Introduction and the main result

This paper is a sequel to [5], where we study the weighted Strichartz estimates for the wave equation without the support condition and self-similar solutions to nonlinear wave equations. The results in [5] are restricted to odd space dimensions, for the proof of the weighted Strichartz estimates depends on an explicit representation formula of the free solutions which holds only for odd space dimensions. The corresponding explicit representation formula for even space dimensions causes a number of serious difficulties when one follows a similar method. One of the difficulties lies in the existence of region of diffusion of waves, where a similar technique seems useless to control the tail of waves which lives inside the light cone and has a singularity on the light cone. It seems technical as far as the difference of representation formula is concerned, while it looks essential because the situation depends heavily on the Huygens principle.

* COE fellow
In this paper we prove the weighted Strichartz estimates for the wave equation in even space dimensions. Let $F$ be a function on $\mathbb{R}^{1+n} = (0, \infty) \times \mathbb{R}^n$ with radial symmetry in space and let $w$ be a solution of the wave equation

\begin{align*}
\Box w &= F, \quad (t, x) \in \mathbb{R}^{1+n}, \\
w|_{t=0} &= \partial_t w|_{t=0} = 0, \quad x \in \mathbb{R}^n,
\end{align*}

where $\Box = \partial_t^2 - \Delta$ is the d’Alembertian with Laplacian $\Delta$ in $\mathbb{R}^n$.

**Theorem 1.** Let $n \geq 2$ be even and let $2 < q < \frac{2(n+1)}{n-1}$. Let $a$ and $b$ satisfy

\begin{align*}
a - b + \frac{n+1}{2} &= \frac{n-1}{2}, \\
\frac{n}{q} - \frac{n-1}{2} < b < \frac{1}{q}.
\end{align*}

Then, there exists a constant $C > 0$ such that

\begin{align*}
\|t^2 - |x|^2|^a w\|_{L^q(\mathbb{R}^{1+n})} \leq C \|t^2 - |x|^2|^b F\|_{L^q(\mathbb{R}^{1+n})}.
\end{align*}

**Remark 1.** (1) Theorem 1 also holds when $n$ is odd. See [5, Lemma 3.1].

(2) A similar estimate to Theorem 1 has been shown by Georgiev-Lindblad-Sogge [2, Theorem 1.4] in odd space dimensions, and they announced that the corresponding even space dimensional cases also hold. In the above theorem their support condition $\text{supp } F \subset \{(t, x); |x| < t\}$ is removed at the cost of an additional lower bound $b > \frac{n}{q} - \frac{n-1}{2}$.

(3) As for the weighted Strichartz estimates without radial symmetry, see [2], [15], which require that the support of $F$ is contained in the light cone.

The proof of Theorem 1 is based on explicit representations of the solution $w$ derived by using the radial symmetry of the inhomogeneous term $F$. Such representation and duality arguments enables us to reduce the weighted Strichartz estimates to the weighted Hardy-Littlewood-Sobolev inequality, as in the odd space dimensional cases [5]. As compared with the odd dimensional cases, the lack of the Huygens principle makes the treatment of the even dimensional cases more difficult. To overcome such difficulties we divide the proof of Theorem 2 into two cases, $2 < q < \frac{2(n-1)}{n-2}, \frac{2(n-1)}{n-2} < q < \frac{2(n+1)}{n-1}$, and in each case we apply a different representation of the solution, which is due to Rammaha [11] and Takamura [14] and to Kubo-Kubota [6], respectively.

Theorem 1 has an application to the existence of self-similar solutions to the Cauchy problem for semilinear wave equations of the form

\begin{align*}
\Box u &= f(u), \quad (t, x) \in \mathbb{R}^n, \\
\phi|_{t=0} &= \varepsilon \phi, \quad \partial_t \phi|_{t=0} = \varepsilon \psi, \quad x \in \mathbb{R}^n,
\end{align*}

where $\varepsilon > 0$ is a small parameter and $f(u)$ is homogeneous of degree $p$ with respect to $u$ and satisfies the estimates

\begin{align*}
|f(u)| &\leq C |u|^p, \\
|f(u) - f(v)| &\leq C(|u|^{p-1} + |v|^{p-1})|u - v|,
\end{align*}