Estimates for the \( \bar{\partial} \)-Neumann problem and nonexistence of \( C^2 \) Levi-flat hypersurfaces in \( \mathbb{C}P^n \^*

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Abstract. Let \( \Omega \) be a pseudoconvex domain with \( C^2 \) boundary in \( \mathbb{C}P^n \), \( n \geq 2 \). We prove that the \( \bar{\partial} \)-Neumann operator \( N \) exists for square-integrable forms on \( \Omega \). Furthermore, there exists a number \( \epsilon_0 > 0 \) such that the operators \( N, \bar{\partial}^*N, \bar{\partial}N \) and the Bergman projection are regular in the Sobolev space \( W^\epsilon(\Omega) \) for \( \epsilon < \epsilon_0 \). The \( \bar{\partial} \)-Neumann operator is used to construct \( \bar{\partial} \)-closed extension on \( \Omega \) for forms on the boundary \( b\Omega \). This gives solvability for the tangential Cauchy-Riemann operators on the boundary. Using these results, we show that there exist no non-zero \( L^2 \)-holomorphic \((p,0)\)-forms on any domain with \( C^2 \) pseudoconcave boundary in \( \mathbb{C}P^n \) with \( p > 0 \) and \( n \geq 2 \). As a consequence, we prove the nonexistence of \( C^2 \) Levi-flat hypersurfaces in \( \mathbb{C}P^n \).

1. Introduction

A main result of this paper is the following:

**Theorem 1.** There exists no \( C^2 \) real Levi-flat hypersurface in \( \mathbb{C}P^n \), \( n \geq 2 \).

A \( C^2 \) real hypersurface \( M \) is called Levi-flat if its Levi-form vanishes on \( M \). Theorem 1 is inspired by two recent papers of Siu [Siu2, Siu3] who proved that there exists no \( C^{5,0} \) Levi-flat hypersurface in \( \mathbb{C}P^n \), \( n \geq 2 \). The required smoothness has been reduced to \( C^4 \) by Iordan [Io]. Nonexistence of real analytic Levi-flat hypersurfaces in \( \mathbb{C}P^n \) was obtained earlier in Lins Neto [LNe] for \( n \geq 3 \).

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Our proof of Theorem 1 depends on the $L^2$ existence and regularity of the $\bar{\partial}$-Neumann problem for pseudoconvex domains with $C^2$ boundary in $\mathbb{C}P^n$ and is much simpler than the work cited above. The $\bar{\partial}$-Neumann problem has been studied extensively for pseudoconvex domains in $\mathbb{C}^n$ or in a Stein manifold (see [FK] or [Hö3]). In contrast, the $\bar{\partial}$-Neumann problem on pseudoconvex domains in $\mathbb{C}P^n$ has not been studied systematically before.

Let $\Omega_1 \subset \subset \mathbb{C}P^n$ be a pseudoconvex domain with $C^2$-smooth boundary $\partial \Omega_1$ and let $\delta(x) = d(x, b\Omega_1)$ be the distance function from $x \in \Omega_1$ to $b\Omega_1$. We call $t_0 = t_0(\Omega_1)$ the order of plurisubharmonicity for the distance function $\delta$ if

$$t_0(\Omega_1) = \sup\{0 < \epsilon \leq 1 | i\partial \bar{\partial}(-\delta^\epsilon) \geq 0 \text{ on } \Omega_1\}.$$ 

In $\mathbb{C}P^n$ with the standard Fubini-Study metric, Ohsawa-Sibony [OS] showed that there exists $0 < t_0(\Omega_1) \leq 1$ for any pseudoconvex domain $\Omega \subset \mathbb{C}P^n$ with $C^2$-smooth boundary using results of Takeuchi [Taku] and [DF1]. The other main result in this paper is the following:

**Theorem 2.** Let $\Omega$ be a pseudoconvex domain with $C^2$-smooth boundary in $\mathbb{C}P^n$ and let $t_0$ be the order of plurisubharmonicity for the distance function $\delta$. Then the $\bar{\partial}$-Neumann operator $N(p,q)$ exists on $L^2(p,q)(\Omega_1)$ where $0 \leq p, q \leq n$. Furthermore, $\bar{\partial}N$, $\partial N$ and the Bergman projection $P$ are exact regular on $W^s(p,q)(\Omega_1)$ for $0 < s < \frac{1}{2}t_0$ with respect to the $W^s(\Omega)$-Sobolev norms.

We give a self-contained treatment of the $\bar{\partial}$-Neumann problem on pseudoconvex domains in $\mathbb{C}P^n$ in Sections 2 and 3.

In Section 4 we use Theorem 2 to study the tangential Cauchy-Riemann equation on the pseudoconvex boundary $M = b\Omega$

$$\bar{\partial}_b u = f \quad \text{in } M, \quad (1.1)$$

where $f$ is a $(p, q)$-form in $M$ satisfying some compatibility condition with $1 \leq q \leq n - 1$. When $n > 2$ and $q < n - 1$, the compatibility condition for equation (1.1) to be solvable is that

$$\bar{\partial}_b f = 0 \quad \text{on } M. \quad (1.2)$$

When $n \geq 2$ and $q = n - 1$, (1.2) is satisfied trivially and the compatibility condition for equation (1.1) is substituted by the following moment condition:

$$\int_M f \wedge \Psi = 0, \quad (1.3)$$

where $\Psi$ is any $\bar{\partial}_b$-closed $(n, 0)$-form on $M$ (see (9.2.12a) of [CS, p216].) In [Siu2, Siu3], existence and regularity of equation (1.1) on a Levi-flat boundary $M$ in $\mathbb{C}P^n$ is studied. The two different compatibility conditions (1.2) and (1.3) underlie the reason why earlier work differentiate $\mathbb{C}P^n$ when $n \geq 3$ and $\mathbb{C}P^2$.

When $M$ is a $C^2$ Levi-flat hypersurface in $\mathbb{C}P^n$, we show that the compatibility condition (1.3) is void by deriving several Liouville type theorems which are of independent interest. In particular, we show that