Traps for reflected Brownian motion

Krzysztof Burdzy*, Zhen-Qing Chen**, Donald E. Marshall***

Department of Mathematics, Box 354350, University of Washington Seattle, WA 98115-4350, USA (e-mail: burdzy@math.washington.edu; zchen@math.washington.edu; marshall@math.washington.edu)

Received: 12 March 2004; in final form: 2 May 2005 / Published online: 16 August 2005 – © Springer-Verlag 2005

Abstract. Consider an open set $D \subset \mathbb{R}^d$, $d \geq 2$, and a closed ball $B \subset D$. Let $\mathbb{E}^x T_B$ denote the expectation of the hitting time of $B$ for reflected Brownian motion in $D$ starting from $x \in D$. We say that $D$ is a trap domain if $\sup_x \mathbb{E}^x T_B = \infty$. A domain $D$ is not a trap domain if and only if the reflecting Brownian motion in $D$ is uniformly ergodic. We fully characterize the simply connected planar trap domains using a geometric condition and give a number of (less complete) results for $d > 2$.

Mathematics Subject Classifications (2000): 60J45, 35P05, 60G17

1. Introduction

In this section, we will limit ourselves to an informal statement of the problem and a brief review of our results. See Section 2 for rigorous statements of the theorems and Section 3 for the proofs.

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be an open connected set with a finite volume and let $X$ be the normally reflected Brownian motion (RBM) on $D$ constructed using Dirichlet form methods (see section 2 for details). Note that $X$ is well-defined for every starting point in $D$ and for $x \in D$ we let $\mathbb{P}^x$ denote the distribution of $X_t$ starting from $X_0 = x$, with corresponding expectation $\mathbb{E}^x$. Let $B \subset D$ be a closed ball with non-zero radius and denote by $T_B = \inf \{t \geq 0 : X_t \in B\}$ the first hitting time of $B$ by $X$. If $\mathbb{E}^x T_B$ is very large for some $x$ then RBM starting from $x$ appears to be trapped near the boundary of $D$. We will say that $D \subset \mathbb{R}^d$, $d \geq 2$, is a trap domain if

* Research partially supported by NSF grant DMS-0303310.
** Research partially supported by NSF grant DMS-0303310.
*** Research partially supported by NSF grant DMS-0201435.
and otherwise $D$ is called a non-trap domain. The definition of a trap domain does not depend on the choice of $B$ (see Lemma 3.3 in the last section). At this point, the reader might like to consult Proposition 2.13, Proposition 2.11, and Figure 3.2 below for some simple examples of trap and non-trap domains.

Our article is mainly devoted to the following problem.

**Problem 1.1.** Find necessary and sufficient geometric conditions for $D$ to be a trap domain.

The notion of a trap domain is closely related to the notion of Markov chain ergodicity (see [MT], Part III). We will make this remark more precise in the next proposition. Let $\|\mu\|_{TV}$ denote the total variation norm of a measure $\mu$. When $\mu = f(x)\,dx$, then $\|\mu\|_{TV} = \int |f(x)|\,dx$. Let $\Pi_D$ denote the uniform probability measure in $D$.

**Proposition 1.2.** Let $D \subset \mathbb{R}^d$ be a connected open set with finite volume. Then the following are equivalent.

(i) $D$ is non-trap.

(ii) $\lim_{t \to \infty} \sup_{x \in D} \|\mathbb{P}^x(X_t \in \cdot) - \Pi_D\|_{TV} = 0$.

(iii) There are positive constants $c_1$ and $c_2$ such that $\sup_{x \in D} \|\mathbb{P}^x(X_t \in \cdot) - \Pi_D\|_{TV} \leq c_1 e^{-c_2 t}$.

Properties (ii) and (iii) are called the uniform ergodicity of reflecting Brownian motion in $D$. The above equivalence is proved for discrete time Markov chains in Theorem 16.0.2 (ii) and (vi) of [MT].

It will be convenient to express Problem 1.1 in purely analytic terms. Let $G(x, y)$ be defined on $(D \setminus B) \times (D \setminus B)$ by

$$
\int_{(D \setminus B) \cap A} G(x, y)\,dy = \mathbb{E}^x \int_0^{T_B} \mathbf{1}_{\{X_t \in A\}}\,dt, \quad A \subset \overline{D},
$$

where $dy$ denotes $d$-dimensional Lebesgue measure. In other words, $G(x, y)$ is the Green function for the domain $D \setminus B$ with the (zero) Neumann boundary conditions on $\partial D$ (in the distributional sense) and (zero) Dirichlet boundary conditions on $\partial B$.

The existence of $G(x, y)$ follows from a result in [Fu] saying that there exists a strictly positive function $p_t(x, y)$ on $(0, \infty) \times D \times D$ such that for every $x \in D$ and $A \subset \overline{D}$,

$$
\mathbb{P}^x(X_t \in A) = \int_{D \cap A} p_t(x, y)\,dy,
$$

(see Section 3.1 below for details). We will call $p_t(x, y)$ the Neumann heat kernel on $D$. From a technical point of view, it is easier to define the Green function with the specified boundary conditions than the corresponding RBM. The condition

$$
\sup_{x \in D \setminus B} \int_{D \setminus B} G(x, y)\,dy = \infty,
$$

(1.2)