On the equivariant formality of Kähler manifolds with torus group actions

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Abstract We consider simply connected compact Kähler manifolds which have a holomorphic action of a torus group. We use the existing equivariant models for rational homotopy to show that these spaces satisfy an equivariant formality condition over the complex numbers.

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1 Introduction

A space is formal if its rational homotopy type is determined by its rational cohomology ring. To make this precise, formality is defined using the category of commutative differential graded algebras (CDGAs). A CDGA is formal if it is quasi-isomorphic to its cohomology ring, regarded as a CDGA with differential $d = 0$. This means that $A$ is formal if there is a zig-zag of quasi-isomorphisms $A \rightarrow A_1 \leftarrow A_2 \rightarrow \cdots \rightarrow H^*(A)$. For CDGAs which are minimal in the sense of Sullivan [5, 16], formality can be described more simply, since the zig-zag of quasi-isomorphisms in this case is equivalent to the existence of a single quasi-isomorphism $A \rightarrow H^*(A)$.

To define formality for spaces, we use the minimal model $\mathcal{M}_X$ of a space $X$ from [5, 16], which is a minimal CDGA. A space $X$ is formal if its minimal model $\mathcal{M}_X$ is formal. The CDGA $\mathcal{M}_X$ encodes all rational homotopy information, including homotopy and homology groups; in particular, the cohomology $H^*(X; \mathbb{Q}) = H^*(\mathcal{M}_X)$. If a space $X$ is formal then there is a quasi-isomorphism $\mathcal{M}_X \rightarrow H^*(\mathcal{M}_X) = H^*(X; \mathbb{Q})$, and we may compute the model $\mathcal{M}_X$ of $X$ as the minimal model for the rational cohomology ring $H^*(X; \mathbb{Q})$. Therefore the rational homotopy type of a formal space can
be recovered from the cohomology ring. Since cohomology is generally more accessible than other invariants, this can make calculations much easier. Several interesting classes of spaces turn out to be formal, and the paper of Deligne et al. [5] which introduced Sullivan’s minimal models used them to show the formality of simply connected compact Kähler manifolds as its primary application.

Equivariantly, where algebraic models exist which describe the rational homotopy type of \( G \)-spaces, they can be used to define an analogous notion of formality. For actions of finite groups, Triantafillou and Fine [7] have shown that simply connected compact Kähler manifolds are equivariantly formal over the complex numbers. In this paper, we extend this result and consider actions of torus groups \( T \). Working with complex coefficients, we prove that simply connected compact Kähler manifolds with holomorphic \( T \)-actions are equivariantly formal. This result uses the definition of equivariant formality from [14], which is based on the algebraic models for \( T \)-spaces developed in [11,12].

Throughout this paper, the \( T \)-spaces under consideration will be compact manifolds with a smooth action of a torus \( T \); in particular, this implies that they are \( T \)-CW complexes, and all orbit spaces and related constructions, such as Borel spaces, are also CW complexes. In addition, compactness assures that all \( T \)-spaces have finitely many orbit types and that the rational cohomology of each fixed point subspace \( X^H = \{ x \in X \mid hx = x \text{ for all } h \in H \} \) is of finite type for all closed subgroups \( H \subseteq T \). We will also assume that all \( T \)-spaces are based, with basepoint fixed by the \( T \)-action, and \( T \)-simply connected in the sense that the connected components of the fixed point subspaces \( X^H \) are all simply connected. These will be standing assumptions in what follows; note that we are not assuming that the fixed point sets are also connected, as needed for the minimal models of [12], but instead are allowing disconnected fixed sets as in [11], as is necessary for the study of Kähler manifolds.

Section 2 gives a quick sketch of the background material on equivariant formality. Section 3 contains the proof of the equivariant formality of compact Kähler \( T \)-manifolds. Section 4 has two simple examples of computations for a linear circle action on complex projective spaces, and Sect. 5 contains the proof of an important but somewhat involved proposition used to translate from the usual equivariant model to a more convenient one in the proof of the main theorem.

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### 2 Equivariant models and formality

The equivariant homotopy type of a space \( X \) with the action of a group \( G \) depends not only on the homotopy type of the space itself but also on the homotopy type of the fixed point subspaces \( X^H \) for all closed subgroups \( H \subseteq G \). Together with the natural inclusions and maps induced by the action of \( G \), these form a diagram of spaces indexed by the closed subgroups of \( G \). Much of equivariant homotopy theory makes use of this diagram, and we often define algebraic invariants by constructing diagrams of algebras reflecting the fixed point data.

In defining equivariant algebraic models for the rational homotopy of spaces with actions of torus groups, we use the diagram category defined in [11]. Because we are dealing with actions of connected torus groups, we do not need the full generality