On commuting polynomial automorphisms of $\mathbb{C}^k$, $k \geq 3$

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Abstract  We characterize the polynomial automorphisms of $\mathbb{C}^3$, which commute with a regular automorphism. We use their meromorphic extension to $\mathbb{P}^3$ and consider their dynamics on the hyperplane at infinity. We conjecture the additional hypothesis under which the same characterization is true in all dimensions. We give a partial answer to a question of S. Smale that in our context can be formulated as follows: can any polynomial automorphism of $\mathbb{C}^k$ be the uniform limit on compact sets of polynomial automorphisms with trivial centralizer (i.e. $C(f) \simeq \mathbb{Z}$)?

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1 Introduction

Complex affine k-space, $\mathbb{C}^k$, is one of the basic objects in complex analysis and geometry. It seems quite hard to give an algebraic description of the group of (polynomial) automorphisms of $\mathbb{C}^k$, when $k \geq 3$. The group of polynomial automorphisms of $\mathbb{C}^k$, $\text{Aut}(\mathbb{C}^k)$, consists of bijective maps:

$$f : (z_1, \ldots, z_k) \in \mathbb{C}^k \rightarrow (f_1(z_1, \ldots, z_k), \ldots, f_k(z_1, \ldots, z_k)) \in \mathbb{C}^k$$


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where \( f_1, \ldots, f_k \in \mathbb{C}[z_1, \ldots, z_k] \). When \( f \) is polynomial and bijective, then the inverse \( f^{-1} \) is a polynomial mapping.

In dimension 2, the algebraic structure of the group of polynomial automorphisms is well known. The result is due to Jung [10]; it was reproved in several different ways [17] and recently also in [14]. Jung’s theorem asserts that the group \( \text{Aut}(\mathbb{C}^2) \) is the amalgamated product of its subgroups \( \mathcal{E} \) and \( \mathcal{A} \) with respect to their intersection \( \mathcal{A} \cap \mathcal{E} \), where the group \( \mathcal{E} \) of \textit{elementary} maps is:

\[
\mathcal{E} = \{(z, w) \mapsto (az + p(w), \beta w + \gamma) : \alpha, \beta, \gamma \in \mathbb{C}, \alpha \beta \neq 0, p \in \mathbb{C}[w]\}
\]

the group \( \mathcal{A} \) of \textit{affine} maps is:

\[
\mathcal{A} = \{(z, w) \mapsto (a_1 z + b_1 w + c_1, a_2 z + b_2 w + c_2) : a_i, b_i, c_i \in \mathbb{C}, a_1 b_2 - a_2 b_1 \neq 0\}
\]

and where \( \mathcal{A} \mathcal{T} \) denote the intersection \( \mathcal{A} \cap \mathcal{E} \), i.e. the group of the automorphisms \textit{affine and triangular}:

\[
\mathcal{A} \mathcal{T} = \{(z, w) \mapsto (a_1 z + b_1 w + c_1, a_2 z + b_2 w + c_2) : a_1, b_1, c_1 \in \mathbb{C}, a_1 b_2 - a_2 b_1 \neq 0\}.
\]

By this structure theorem, each automorphism \( \varphi \in (\text{Aut}(\mathbb{C}^2) \setminus \mathcal{A} \mathcal{T}) \) can be written as a composition of elementary and affine automorphisms. In 1989 Friedland and Milnor [8] proved that any polynomial automorphism of \( \mathbb{C}^2 \) is conjugated either to an elementary map or to a finite composition of Hénon maps \( h_j \) defined as follows

\[
h_j(z, w) = (p_j(z) - a_j w, z), \quad a_j \in \mathbb{C},
\]

where \( \deg(p_j) \geq 2 \). We denote by \( \mathcal{H} \) the semigroup generated by Hénon maps.

On the other hand, the algebraic structure of \( \text{Aut}(\mathbb{C}^k) \), \( k \geq 3 \), is poorly understood even if the Nagata Conjecture has been recently proved [22,23]. Recently Shestakov and Umirbaev [22,23] have proved that tame and wild polynomial automorphisms of \( \mathbb{C}^3 \) are algorithmically recognizable. The following Nagata automorphism in \( \text{Aut}(\mathbb{C}[x, y, z]) \),

\[
\begin{align*}
\sigma(x) &= x + (x^2 - yz)z, \\
\sigma(y) &= y + 2(x^2 - yz)x + (x^2 - yz)^2 z, \\
\sigma(z) &= z
\end{align*}
\]

provides a candidate of such wild automorphisms.

We recall now some general facts. Let \( z = (z_1, \ldots, z_k) \) be affine coordinates in \( \mathbb{C}^k \) and let \( [z : t] = [z_1 : \cdots : z_k : t] \) be corresponding homogeneous coordinates in \( \mathbb{P}^k \), then the hyperplane at infinity \( \Pi_{\infty} \) has equation \( \{t = 0\} \).

Each polynomial automorphism \( f \) of \( \mathbb{C}^k \) can be considered as a birational map \( \overline{f} \) of \( \mathbb{P}^k \). We will denote, respectively, \( I^+_f \) and \( I^-_f \) the indeterminacy subsets of \( \overline{f} \) and of \( \overline{f}^{-1} \). These are two analytic and algebraic subsets of complex codimension at least 2 in \( \mathbb{P}^k \), contained in \( \Pi_{\infty} \). In the sequel we are going to write \( f \) instead of \( \overline{f} \). In a point \( p \in I^+_f \) it is possible to define the \textit{blow-up} of \( f \) in \( p \) which is the set

\[
B^f_p = \bigcap_{\epsilon > 0} \overline{f}(\mathbb{P}(p, \epsilon) \setminus I^+_f)
\]

In other words it is the fiber over \( p \) in the closure of the graph of \( f \) and it is an analytic subset of \( \Pi_{\infty} \) of dimension \( h \) with \( 1 \leq h \leq (k - 1) \). We will say, [20], that \( f \) is an \textit{algebraically stable} polynomial automorphism if and only if \( \overline{f}^n([z : 0]) \setminus I^+_f \) is not...