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The moduli space of regular stable maps

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Abstract We prove that the moduli space of regular stable maps in a complex manifold admits a natural complex orbifold structure. Our proof is based on Hardy decompositions and Fredholm intersection theory.

1 Introduction

This paper is a sequel to [4]. It studies the moduli space of stable maps whereas [4] studied the moduli space of stable marked nodal Riemann surfaces. The latter can be considered as a special case of the former by taking the target manifold M to be a point. In both cases the moduli space is the orbit space of a groupoid where the objects are compact surfaces with additional structure. (We think of a map from a surface to another manifold as a structure on the surface.) In both cases the difficulty is that to achieve compactness of this moduli space it is necessary to include objects whose underlying surfaces are not homeomorphic.

Here we study only that part of the moduli space of stable maps which can be represented by regular stable maps. Only by restricting attention to regular stable maps can we hope to construct an orbifold structure. We also limit attention to target manifolds M which are integrable complex and not just almost complex.

As in [4] we make heavy use of "Hardy decompositions". The idea is to decompose a Riemann surface Σ into two surfaces Σ' and Σ'' intersecting in their common boundary Γ. A holomorphic map from Σ into a complex manifold M is uniquely determined by its

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restriction to Γ and so the space of all such holomorphic maps can be embedded into the space $\mathcal{V}$ of smooth maps from Γ to M. In this way we identify the holomorphic maps with $\mathcal{V}' \cap \mathcal{V}''$ where $\mathcal{V}'$ and $\mathcal{V}''$ are the maps from Γ to M which extend holomorphically to Σ' and Σ'' respectively. (In the case where Σ is the Riemann sphere, $M = \mathbb{C} \cup \{\infty\}$, and Γ is the equator, $\mathcal{V}'$ would consist of those maps whose negative Fourier coefficients vanish and $\mathcal{V}''$ would consist of those maps whose positive Fourier coefficients vanish. Hence the name *Hardy decomposition*.) The importance of this construction becomes clear when we consider a parameterized family $\{\Sigma_b\}_{b\in B}$ of Riemann surfaces. By judiciously choosing the decomposition we can arrange that the one dimensional manifolds Γ_b are all diffeomorphic, even though the manifolds Σ'_b are not all homeomorphic. Then we identify the various Γ_b with a disjoint union Γ of circles. Under suitable hypotheses we are able to represent the holomorphic maps from Σ_b to M (for varying b) as a submanifold of the manifold of smooth maps from $\Gamma \cong \partial\Sigma'_b = \partial\Sigma''_b$ to M.

Our theorems led to a theory of Fredholm triples in Sect. 6. These are triples (X, X', X'') where X is a Hilbert manifold and X', X'' are Hilbert submanifolds such that $T_xX' \cap T_xX''$ and $T_xX/(T_xX' + T_xX'')$ are finite dimensional for every $x \in X' \cap X''$. We prove a finite dimensional reduction theorem for morphisms of such triples. We hope this theory has separate interest.

In Sect. 8 we show that the orbifold topology is the same as the well known topology of Gromov convergence.

Naming the additional structures which occur in this paper as opposed to [4] caused us to exhaust the Latin and Greek alphabets. Accordingly we have changed notation somewhat. For example, the aforementioned decomposition $\Sigma = \Sigma' \cup \Sigma''$ was $\Sigma = \Delta \cup \Omega$ in [4]. We also use the following notations

$$\begin{aligned} \mathsf{g} &:= \text{arithmetic genus of } \Sigma/\nu, \\ \mathsf{n} &:= \text{number of marked points}, \\ \mathsf{k} &:= \text{number of nodal points}, \\ \mathsf{a} &:= \text{complex dimension of } A, \\ \mathsf{b} &:= \text{complex dimension of } B, \\ \mathsf{m} &:= \text{complex dimension of } M. \end{aligned}$$

We have used the `\mathsf` font for these integers so that we can write $a \in A$, $b \in B$ for the elements. We will also use the symbol d to denote a homology class in $H_2(M; \mathbb{Z})$.

2 Stable maps

2.1 Throughout let (M, J) be a complex manifold without boundary. A **configuration** in M is a tuple (Σ, s_*, ν, j, v) where (Σ, s_*, ν, j) is a marked nodal Riemann surface (see [4, Sect. 3]) whose quotient Σ/ν is connected and $v : \Sigma \to M$ is a smooth map satisfying the nodal conditions

$$\{x, y\} \in \nu \implies v(x) = v(y).$$

Thus v descends to the quotient Σ/ν and we write $v : \Sigma/\nu \to M$ for a smooth map $v : \Sigma \to M$ satisfying the nodal conditions. We say that the configuration has **type** (g, n) if the marked nodal surface (Σ, s_*, ν) has type (g, n) in the sense of [4, Definition 3.7] and that