Domestic canonical algebras and simple Lie algebras

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Abstract For each simply-laced Dynkin graph $\Delta$ we realize the simple complex Lie algebra of type $\Delta$ as a quotient algebra of the complex degenerate composition Lie algebra $L(A)_1^C$ of a domestic canonical algebra $A$ of type $\Delta$ by some ideal $I$ of $L(A)_1^C$ that is defined via the Hall algebra of $A$, and give an explicit form of $I$. Moreover, we show that each root space of $L(A)_1^C/I$ has a basis given by the coset of an indecomposable $A$-module $M$ with root easily computed by the dimension vector of $M$.

Keywords Simple Lie algebras · Hall algebras · Canonical algebras

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Introduction

Let $A$ be a finite-dimensional algebra over a finite field $k$ with $q$ elements, and consider the free abelian group $\mathcal{H}(A)$ with basis the isoclasses of finite $A$-modules. Then by Ringel [24] $\mathcal{H}(A)$ turns out to be an associative ring with identity, called the integral Hall algebra of $A$, with respect to the multiplication whose structure constants are given by the numbers of filtrations of modules with factors isomorphic to modules that are multiplied (see Definition 2.1). The free abelian subgroup $L(A)$ of $\mathcal{H}(A)$ with basis the isoclasses of finite indecomposable $A$-modules becomes a Lie subalgebra modulo $q - 1$ whose Lie bracket is given by the commutator of the Hall multiplication. We call this Lie bracket the Hall commutator. It would be interesting to realize all types of simple (complex) Lie algebras using this Hall commutator.

Along this line, Ringel [25] realized the positive part of the simple Lie algebra $\mathfrak{g}(\Delta)$ for each Dynkin type $\Delta$. Further Peng and Xiao [18] realized all types of simple Lie algebras by...
the so-called root categories of finite-dimensional representation-finite hereditary algebras. But the Lie bracket was not completely given by the Hall commutator, because the root category \( \mathcal{R} \) provides only the positive and the negative parts. The Cartan subalgebra \( \mathfrak{h} \) was given by a subgroup of the Grothendieck group of \( \mathcal{R} \) over the field \( \mathbb{Q} \) of rational numbers. The Hall commutator was used to define the Lie bracket only inside \( \mathcal{R} \), and when the bracket should not be closed in \( \mathcal{R} \), namely when we deal with an indecomposable object \( X \) in \( \mathcal{R} \) of a root \( \alpha \) and an indecomposable object \( Y \) in \( \mathcal{R} \) of the root \(-\alpha\), the definition of the bracket \([X, Y]\) was changed in order to have \([X, Y] \in \mathfrak{h}\). In [1] we succeeded to realize general linear algebras and special linear algebras (see also Iyama [13]) by the Hall commutator defined on cyclic quiver algebras. In this realization also the Cartan subalgebra was naturally provided together with the positive and the negative parts. In [2] we gave a way how to realize all types of simple Lie algebras by the Hall commutator using tame hereditary algebras, in particular we gave an explicit realization of simple Lie algebras of type \( \mathfrak{D}_n \).

However this realization needed some rational constants to define a necessary ideal of the Lie algebra. In this paper we give another realization by using domestic canonical algebras. Here we do not use a surjective Lie algebra homomorphism from an affine Lie algebra [14] that was an essential tool in [2]. In the realization using a tame hereditary algebra we had to choose some orientation for the quiver of the algebra. But if we use a canonical algebra we are free from choosing an orientation of the quiver of the algebra except for the \( A_n \) case because the orientation is given from the beginning. For simplicity we deal only with simply-laced cases. Non-simply-laced cases may be treated using the generalized definition of canonical algebras by Ringel [23]. We expect that the same approach works to realize affine Kac-Moody algebras by using tubular canonical algebras instead of domestic ones. (In fact, Zhengxin Chen is carrying out this plan, the primary version [5] contained a similar error as in the first version of this paper.) It should be pointed out that in the realization using canonical algebras the preprojective (resp. preinjective) component contains only basis vectors of the positive (resp. negative) part (see Remarks 4.7 and 8.1), in contrast, in the realization using hereditary (non-canonical) algebras the preprojective component and the preinjective component contain basis elements of both positive and negative parts. Finally we mention that there is a possibility to construct representations of simple Lie algebras by the form of our realization using infinite-dimensional modules as done in [13].

The first version contained a serious error that the constructed Lie algebra may turn out to be zero because the relations required on it was too much. This problem was fixed in the present version.

The paper is organized as follows. After preliminaries in Sect. 1 we collect necessary facts on Lie algebra constructions using Hall algebras, and domestic canonical algebras in Sects. 2 and 3, respectively. In Sect. 4 we state our main theorem, and Sect. 5 is devoted to preparations of our proof of the main theorem. We give a proof of the main theorem in Sect. 6. We next examine root spaces of the Lie algebra constructed here to prove the remaining theorem in Sect. 7. Finally in the last section we exhibit an example of basis vectors of the realization of simple Lie algebra of type \( \mathfrak{D}_5 \), and an example that shows an error in the first version of the paper.

1 Preliminaries

1.1 Notation

Throughout this paper \( k \) is a finite field of cardinality \( q \geq 3 \). When we deal with domestic canonical algebras of type \( \mathfrak{E}_8 \) (see Sect. 3.1 for definition) we assume that \( \text{char} \, k \neq 2 \). For