Continuity of the Complex Monge–Ampère operator on compact Kähler manifolds

Yang Xing

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Abstract We prove several approximation theorems of the complex Monge–Ampère operator on a compact Kähler manifold. As an application we prove the Cegrell type theorem on a complete description of the range of the complex Monge–Ampère operator in the class of \(\omega\)-plurisubharmonic functions with vanishing complex Monge–Ampère mass on all pluripolar sets. As a by-product we obtain a stability theorem of solutions of complex Monge–Ampère equations.

Keywords Complex Monge–Ampère operator · Compact Kähler manifold

Mathematics Subject Classification (2000) Primary 32W20 · 32Q15

1 Introduction

Let \(X\) be a compact connected Kähler manifold of complex dimension \(n\), equipped with the fundamental form \(\omega\) given in local coordinates by

\[
\omega = \frac{i}{2} \sum_{\alpha, \beta} g_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta,
\]

where \((g_{\alpha \beta})\) is a positive definite Hermitian matrix and \(d\omega = 0\). The smooth volume form associated to this Kähler metric is given by the \(n\)th wedge product \(\omega^n\). Denote by \(PSH(X, \omega)\) the set of upper semi-continuous functions \(u : X \rightarrow \mathbb{R} \cup \{-\infty\}\) such that \(u\) is integrable in \(X\) with respect to the volume form \(\omega^n\) and \(\omega + dd^cu \geq 0\) on \(X\). Functions in \(PSH(X, \omega)\) are called \(\omega\)-plurisubharmonic functions, which are defined on the whole \(X\) and locally given by the sum of a true plurisubharmonic function and a smooth function. Following the fundamental work of Bedford and Taylor [5], we know that the complex Monge–Ampère operator \((\omega + dd^c)^n\) is well-defined for all bounded \(\omega\)-plurisubharmonic functions in \(X\).
By the Stokes theorem we always have \( \int_X (\omega + dd^c u)^n = \int_X \omega^n \). It is also known that the complex Monge–Ampère operator \((\omega + dd^c u)^n\) does not make sense for all functions in \( PSH(X, \omega) \), see the example of Kiselman [15]. On the other hand, Cegrell [8, 9] introduced several classes of unbounded plurisubharmonic functions in hyperconvex domains in \( \mathbb{C}^n \) for which the complex Monge–Ampère operator is well-defined. See also Blocki’s important contributions on the definition of the complex Monge–Ampère operator [1, 2]. This theory was recently developed by Guedj and Zeriahi [13] to compact Kähler manifolds. The complex Monge–Ampère operator is extremely useful in Kähler geometry. In 1978, Yau confirmed the famous Calabi conjecture in algebraic geometry by solving the following complex Monge–Ampère equations on compact Kähler manifolds.

**Theorem A** [24] If \( \mu \) is a smooth volume form, then there exists a (unique) smooth function \( u \) in \( PSH(X, \omega) \) such that

\[
(\omega + dd^c u)^n = \mu \quad \text{and} \quad \sup_X u = 0.
\]

Theorem A gives the existence of a Kähler metric with any prescribed volume form on a compact Kähler manifold, which has great consequence in differential geometry. Later, Kolodziej [17, 18] solved the complex Monge–Ampère equation in \( PSH(X, \omega) \cap C(X) \) for \( \mu = f \omega^n \), where \( \mu(X) = \int_X \omega^n \) and \( 0 \leq f \in L^p(X) \) with \( \int_X |f|^p \omega^n < \infty \) and \( p > 1 \). Following Cegrell’s work [8, 9], Guedj and Zeriahi [13] introduced the class \( \mathcal{E}(X, \omega) \) of functions \( u \) in \( PSH(X, \omega) \) such that \( \int_X (\omega + dd^c u)^n = \int_X \omega^n \). This class includes all bounded \( \omega \)-plurisubharmonic functions in \( X \) and is the largest class of \( \omega \)-plurisubharmonic functions on which the complex Monge–Ampère operator is well-defined and the comparison principle is valid. They gave a complete description of the range of the complex Monge–Ampère operator in \( \mathcal{E}(X, \omega) \). In this paper, we obtain several approximation theorems of the complex Monge–Ampère operator in \( \mathcal{E}(X, \omega) \). We prove the following approximation theorem.

**Theorem 1** If \( u_j, u \in \mathcal{E}(X, \omega) \) are such that \( u_j \rightharpoonup u \) in the capacity \( \text{Cap}_\omega \) on \( X \), then \( (\omega + dd^c u_j)^n \to (\omega + dd^c u)^n \) weakly in \( X \).

As an application of our approximation theorems we also prove the Cegrell type theorem in the following way: locally applying a well-known result of Cegrell one can easily construct a subsolution of the complex Monge–Ampère equation, and then by means of such a subsolution we find a solution of the equation.

We moreover study stability of solutions of complex Monge–Ampère equations. For two smooth functions \( u \) and \( v \) in \( X \), Calabi [11] proved that if \( (\omega + dd^c u)^n = (\omega + dd^c v)^n \) and \( \max_X u = \max_X v = 0 \) then \( u = v \) in \( X \). Calabi’s uniqueness theorem is an important fact and has been studied in [3, 6, 18]. Now for the subclass \( \mathcal{E}^1(X, \omega) \) of functions \( u \) in \( \mathcal{E}(X, \omega) \) which are integrable on \( X \) with respect to \( (\omega + dd^c u)^n \), we have

**Theorem 8** (Stability theorem) Let \( \mu \) be a finite positive Borel measure \( \mu \) vanishing on all pluripolar subsets of \( X \). Suppose that \( u, u_j \in \mathcal{E}^1(X, \omega) \) with \( \max_X u = \max_X u_j = 0 \) are such that \( (\omega + dd^c u)^n \leq \mu \) and \((\omega + dd^c u_j)^n \leq \mu \) for all \( j \). Then \( u_j \rightharpoonup u \) in \( L^1(X) \) if and only if \( (\omega + dd^c u_j)^n \to (\omega + dd^c u)^n \) weakly in \( X \).

An analogous version for uniformly bounded \( \omega \)-plurisubharmonic functions in a compact Kähler manifold has been studied by Kolodziej in [19]. See also [12, 21] for functions in bounded domains in \( \mathbb{C}^n \).

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