Moufang sets of type $F_4$

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Abstract We give an explicit description of the Moufang sets of type $F_4$, i.e. the buildings arising from the simple algebraic groups of absolute type $F_4$ and relative rank one, over an arbitrary field. We use octonion planes and certain polarities to find this description, and we rely on the theory of Albert algebras. We also determine the automorphism groups of the corresponding exceptional unitals, thereby completing the program of J. Tits for these non-abelian Moufang sets. In particular we prove that every automorphism of that unital is induced by a collineation of the ambient projective plane.

Keywords Moufang set · $F_4$ · Unital · Octonion algebra · Albert algebra

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Secondary 17C40 · 51E15

1 Introduction

Moufang sets were introduced by Jacques Tits in [19] as an axiomization of the isotropic simple algebraic groups of relative rank one, and they are, in fact, the buildings corresponding to these algebraic groups, together with some of the group structure (which comes from the root groups of the algebraic group). In this way, the Moufang sets are a powerful tool to study these algebraic groups.
Formally speaking, a Moufang set is a set $X$ together with a collection of groups $(U_x \leq \text{Sym}(X))_{x \in X}$, such that each $U_x$ acts regularly on $X \setminus \{x\}$, and such that $U_x^\varphi = U_{x\varphi}$ for all $\varphi \in G^\dagger := \langle U_x \mid x \in X \rangle$. The groups $U_x$ are called the root groups of the Moufang set, and the group $G^\dagger$ is called its little projective group. An automorphism of the Moufang set is an element $\psi \in \text{Sym}(X)$ such that conjugation by $\psi$ restricted to $U_x$ is an isomorphism from $U_x$ to $U_{x\psi}$, for each $x \in X$. For the Moufang sets arising from algebraic groups of $k$-rank 1, the set $X$ is the set of all minimal $k$-parabolics, and each $U_x$ is the root subgroup of $x$ (or, equivalently, the unipotent radical of $x$) with respect to a fixed maximal $k$-split torus. The Moufang set of an algebraic group is essentially equivalent to the algebraic group itself; more precisely, the little projective group of the Moufang set is the adjoint representation of the algebraic group.

The classical algebraic groups are very well understood, and the corresponding Moufang sets also have a satisfying and useful description. In contrast, the exceptional groups are much less understood, in particular from an elementary point of view. In this paper, we focus on the algebraic groups of absolute type $F_4$ and relative rank one (i.e. those of type $F_{4,1}^{21}$ in the notation of [17]). Of course, these groups are known to arise as the automorphism group of certain Albert algebras, but this point of view is often too indirect to be useful. Our aim is to give an elementary description of these Moufang sets, and we immediately illustrate its usefulness by completing Tits’ program for these groups (see below).

It was known before (see [11]) that the real algebraic group $G$ of type $F_{4,1}^{21}$ and relative $\mathbb{R}$-rank one, is isomorphic to a centralizer subgroup $C$ of the polarity $\pi$ of the real octonion projective plane $\mathbb{P}^2(\mathbb{O})$ associated with the standard involution of the real octonions $\mathbb{O}$. Moreover, both points of view define in a natural way a Moufang set as follows. On the one hand, let $B \leq G$ be a Borel subgroup of $G$ and let $U$ be the corresponding unipotent radical. Then the system $(B \setminus G, (U^\varphi)_{B \varphi \in B(G)})$ defines a Moufang set. On the other hand, the group $C$ acts on the set $X$ of incident point-line pairs of $\mathbb{O}$ fixed under the polarity $\pi$. For any such pair $P \in X$, the intersection $V_P$ of $C$ with the unipotent radical of the Borel subgroup corresponding with $P$ and related to the algebraic group of relative rank two, defined by the automorphism group of the octonion projective plane $\mathbb{P}^2(\mathbb{O})$, acts sharply transitively on $X \setminus \{P\}$. The corresponding structure $(X, (V_P)_{P \in X})$ is a Moufang set which is isomorphic to $(B \setminus G, (U^\varphi)_{B \varphi \in B(G)})$.

In this paper, we generalize this fact to an arbitrary field of arbitrary characteristic (explicitly allowing characteristic 2), and we use this fact to give an elementary description of these Moufang sets (only using an octonion division algebra over the given field). Not surprisingly, we rely on the theory of Albert algebras to obtain this result.

As an application, we complete Jacques Tits’ program on algebraic groups or relative rank one and of type $BC_1$ (i.e. with non-abelian root groups) for these particular Moufang sets. This program consists of determining the automorphism groups of the associated geometries. These geometries are defined as follows. Let $(X, (U_x \mid x \in X))$ be a Moufang set defined by an algebraic group of relative rank one and of type $BC_1$. Then the center $Z(U_x)$ of $U_x$ coincides with the commutator subgroup $U_x'$ and we can consider the geometry $(X, B)$, with

$$B = \{\{x\} \cup yU'_x : x, y \in X, x \neq y\},$$

with $yU'_x$ the orbit of $y$ under the action of $U'_x$. The conjecture of Tits, yielding a Fundamental Theorem for these geometries, states that $\text{Aut}(X, B)$ is precisely equal to $\text{Aut}(X, (U_x \mid x \in X))$. That is exactly what we prove for the algebraic groups of type $F_{4,1}^{21}$. In fact, our proof gives the extra information that every automorphism of the geometry $(X, B)$ arises from a unique collineation of $\mathbb{P}^2(\mathbb{O})$, which is a most satisfying situation.