Support varieties, AR-components, and good filtrations

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Received: 6 November 2008 / Accepted: 19 September 2009 / Published online: 13 October 2009
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Abstract Let $G$ be a reductive group, defined over the Galois field $\mathbb{F}_p$ with $p$ being good for $G$. Using support varieties and covering techniques based on $G_r T$-modules, we determine the position of simple modules and baby Verma modules within the stable Auslander–Reiten quiver $\Gamma_{1s}(G_r)$ of the $r$th Frobenius kernel of $G$. In particular, we show that the almost split sequences terminating in these modules usually have an indecomposable middle term. Concerning support varieties, we introduce a reduction technique leading to isomorphisms

$$V_{G_r}(Z_r(\lambda)) \cong V_{G_{r-d}}(Z_{r-d}(\mu))$$

for baby Verma modules of certain highest weights $\lambda, \mu \in X(T)$, which are related by the notion of depth.

Mathematics Subject Classification (2000) Primary 16G70; Secondary 17B50

0 Introduction

In the representation theory of finite-dimensional self-injective algebras, the stable Auslander–Reiten quiver has proven to be an important homological invariant, which has been studied for group algebras of finite groups, reduced enveloping algebras of restricted Lie algebras and distribution algebras of infinitesimal group schemes. In the classical context of Frobenius kernels of reductive groups, the relevant algebras are usually of wild representation type, rendering a classification of their indecomposable modules a hopeless task. This fact notwithstanding, one does have a fairly good understanding of the connected components of the corresponding stable Auslander–Reiten quiver. It is therefore of interest to relate this

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information to certain classes of indecomposable modules, such as simple modules, Weyl modules or Verma modules, and to identify the position of the latter within the AR-quiver. The main problem in this undertaking is the lack of a suitable presentation of the underlying algebras, as required by the techniques of abstract representation theory. On the other hand, the theory of rank varieties and support varieties has seen considerable progress over the last years, so that one can hope to exploit these tools in the aforementioned context.

In continuation of work begun in [22], we study in this article those Auslander–Reiten components of the algebras $\text{Dist}(G_r)$ that contain simple modules or baby Verma modules. The underlying algebra $\text{Dist}(G_r)$ consists of the distributions of the $r$th Frobenius kernel of the smooth reductive group scheme $G$, defined over an algebraically closed field of positive characteristic $p$. The case $r = 1$, which pertains to restricted enveloping algebras of reductive Lie algebras, was settled in [22] by means of a detailed analysis of nilpotent orbits in rank varieties, leading to the consideration of groups of types $\text{SL}(2)_1 \times \text{SL}(2)_1$, $\text{SL}(3)_1$ and $\text{SO}(5)_1$. For $r > 1$, rank varieties are less tractable and the structure of the cohomology rings defining support varieties is more complicated. We address these problems by passage to $G_r T$-modules, as described below. With regard to Auslander–Reiten theory, our main results can roughly be summarized as follows:

**Theorem** Let $G$ be a smooth reductive group scheme, defined over $\mathbb{F}_p$. Given a character $\lambda \in X(T)$ and $r \geq 1$, the following statements hold:

1. The simple $G_r$-module $L_r(\lambda)$ is either projective or it belongs to an AR-component of tree class $\tilde{A}_{12}$ or $A_\infty$. In the latter case, the middle term of the almost split sequence terminating in $L_r(\lambda)$ is indecomposable.

2. The baby Verma module $Z_r(\lambda)$ is either projective or it belongs to an AR-component of tree class $A_\infty$, with the middle term of the almost split sequence terminating in $Z_r(\lambda)$ being indecomposable.

The approach chosen in this article differs from that of [22] by the systematic use of Jantzen’s category $\text{mod} \ G_r T$ of $G_r T$-modules, which affords almost split sequences. Exploiting the natural ordering on the weights $X(T)$, we first study AR-components for $G_r T$-modules, and then use covering properties of the forgetful functor $\bar{\text{F}} : \text{mod} \ G_r T \rightarrow \text{mod} \ G_r$ to obtain information on the corresponding $G_r$-modules. The second major advantage of working in $\text{mod} \ G_r T$ is its tractability with respect to coverings $\tilde{G} \rightarrow G$, ultimately allowing us to bring Steinberg’s tensor product theorem to bear. Aside from these technical aspects, the highest weight category $\text{mod} \ G_r T$ possesses the much studied subcategory $\mathcal{F}(\Delta)$ of $\tilde{Z}_r$-filtered modules. According to Ringel’s seminal work [46], the corresponding categories of $\Delta$-good modules over finite-dimensional quasi-hereditary algebras afford relative almost split sequences. It turns out that in our context these are merely the AR-sequences of the ambient Frobenius category $\text{mod} \ G_r T$.

Our paper is organized as follows. After a preliminary section, we determine in Sect. 2 the location of the simple $G_r$-modules. For finite groups and restricted enveloping algebras the corresponding problems were studied in [36–39] and [16], respectively. In fact, since the algebra $\text{Dist}(G_r)$ is symmetric, Kawata’s results [37], that were inspired by the modular representation theory of finite groups, can be brought to bear. With the exception of simple modules belonging to blocks of tame representation type, the AR-components containing simple modules are of type $\mathbb{Z}[A_\infty]$, with the simple vertex having only one predecessor (see part (1) of the above Theorem).

In Sect. 3 we recall basic results from [19] concerning the AR-Theory of the Frobenius category $\text{mod} \ G_r T$. As a first application, we exploit coverings $\tilde{G} \rightarrow G$ to show that