Classification of hypersurfaces with constant Möbius curvature in $S^{m+1}$

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Abstract Let $x : M^m \rightarrow S^{m+1}$ be an $m$-dimensional umbilic-free hypersurface in an $(m+1)$-dimensional unit sphere $S^{m+1}$, with standard metric $I = dx \cdot dx$. Let $II$ be the second fundamental form of isometric immersion $x$. Define the positive function $\rho = \sqrt{\frac{m}{m-1}} \|II - \frac{1}{m} tr(II)I\|$. Then positive definite $(0,2)$ tensor $g = \rho^2 I$ is invariant under conformal transformations of $S^{m+1}$ and is called Möbius metric. The curvature induced by the metric $g$ is called Möbius curvature. The purpose of this paper is to classify the hypersurfaces with constant Möbius curvature.

Keywords Möbius metric · Constant sectional curvature · Möbius flat hypersurfaces · Möbius deformable hypersurfaces · Curvature-spiral

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1 Introduction

Let \( x : M^m \to \mathbb{S}^{m+1} \) be an \( m \)-dimensional umbilic-free hypersurface, with the standard metric \( I = dx \cdot dx \), in an \((m+1)\)-dimensional unit sphere \( \mathbb{S}^{m+1} \). Let \( II \) be the second fundamental form and \( H \) the mean curvature of \( x \). Define \( \rho^2 = \frac{m}{m-1} \| II - HI \|^2 \), then positive definite 2-form \( \mathbf{g} = \rho^2 I \) is invariant under conformal group (or Möbius group) of \( \mathbb{S}^{m+1} \) and is called Möbius metric of \( x \). The curvature tensor \( R \) induced by the metric \( \mathbf{g} \) is called Möbius curvature of \( x \). Let \( R_I \) denote the curvature tensor with respect to the metric \( I \).

It is well-known that under a conformal change \( I \to \mathbf{g} = \rho^2 I \) of the metric, the relationship between \( R \) and \( R_I \) is as follows:

\[
\rho^{-2} R(X_1, X_2, X_3, X_4) = R_I(X_1, X_2, X_3, X_4) - \left( (\text{Hess}(\log \rho) - d \log \rho \otimes d \log \rho) \otimes I + I \otimes (\text{Hess}(\log \rho) - d \log \rho \otimes d \log \rho) + \| \nabla \log \rho \|^2 I \otimes I \right)(X_1, X_3; X_2, X_4)
+ \left( (\text{Hess}(\log \rho) - d \log \rho \otimes d \log \rho) \otimes I + I \otimes (\text{Hess}(\log \rho) - d \log \rho \otimes d \log \rho) + \| \nabla \log \rho \|^2 I \otimes I \right)(X_2, X_3; X_1, X_4),
\]

where \( X_r (1 \leq r \leq 4) \) is the tangent vector field of \( M^m \), \( \nabla \) and \( \text{Hess} \) denote the gradient operator and Hessian of \( I \), respectively. Since \( R \) is induced by \( \mathbf{g} \), \( R \) is also invariant under the conformal transformations of \( \mathbb{S}^{m+1} \). By using Gauss equation of hypersurface \( x \):

\[
R_I(X_1, X_2, X_3, X_4) = (II \otimes II + I \otimes I)(X_1, X_3; X_2, X_4)
-(II \otimes II + I \otimes I)(X_2, X_3; X_1, X_4),
\]

we can write

\[
R(X_1, X_2, X_3, X_4) = (\mathbf{B} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{g} + \mathbf{g} \otimes \mathbf{A})(X_1, X_3; X_2, X_4)
-(\mathbf{B} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{g} + \mathbf{g} \otimes \mathbf{A})(X_2, X_3; X_1, X_4),
\]

where

\[
\mathbf{B} = \rho (II - HI),
\]

\[
\mathbf{A} = -(\text{Hess}(\log \rho) - d \log \rho \otimes d \log \rho - HII) - \frac{1}{2} (\| \nabla \log \rho \|^2 - 1 + H^2) I.
\]

Equation (1.2) shows that the curvature tensor \( R \) induced by \( \mathbf{g} \) can be expressed by two Möbius invariants \( \mathbf{B} \) and \( \mathbf{A} \) which are called the Möbius second fundamental form and Blaschke tensor of the hypersurface in \( \mathbb{S}^{m+1} \), respectively. We use \( K(p, \sigma) \) \(( p \in M^m, \sigma \) is a 2-dimensional subspace of \( T_p M^m \)) to denote the sectional curvature of \( R \). \( K(p, \sigma) \) is called the Möbius sectional curvature of \( x \) as it is invariant under the Möbius group of \( \mathbb{S}^{m+1} \). In a point of view of conformal geometry, one of the basic questions in the differential geometry of hypersurfaces is to classify all hypersurfaces in \( \mathbb{S}^{m+1} \) with constant Möbius sectional curvature \( K \) up to Möbius transformations. The purpose of this work is to answer the above basic question. In this paper, for the case of \( m > 3 \), we complete the classification of the \( m \)-dimensional hypersurfaces in \( \mathbb{S}^{m+1} \) with constant Möbius sectional curvature \( K \). We will use the Möbius geometry method of submanifolds which was established by Wang [19].

For the purpose to make our main result intuitive, we use the following notations: \( \mathbb{R}^{m+3}_1 \) denotes Lorentz space with the inner product \( \langle \cdot, \cdot \rangle \) given by

\[
\langle Y, Z \rangle = -y_0z_0 + y_1z_1 + \cdots + y_mz_{m+2}.
\]