Homological mirror symmetry for singularities of type D

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Abstract We prove homological mirror symmetry for Lefschetz fibrations obtained as Sebastiani–Thom sums of polynomials of types A or D. The proof is based on the behavior of the Fukaya category under Sebastiani–Thom summation of a polynomial of type D.

1 Introduction

Let \( n \) be a positive integer. An invertible \( n \times n \)-matrix \( A = (a_{ij})_{i,j=1}^{n} \) with integer components defines a polynomial \( W \in \mathbb{C}[x_1, \ldots, x_n] \) by

\[
W = \sum_{i=1}^{n} x_1^{a_{i1}} \cdots x_n^{a_{in}}.
\]

Note that non-zero coefficients of \( W \) can be absorbed by rescaling \( x_i \). A polynomial obtained in this way is called an invertible polynomial if it has an isolated critical point at the origin. The quotient ring \( R = \mathbb{C}[x_1, \ldots, x_n]/(W) \) is naturally graded by the abelian group \( L \) generated by \( n+1 \) elements \( \vec{x}_i \) and \( \vec{c} \) with relations

\[
a_{i1} \vec{x}_1 + \cdots + a_{in} \vec{x}_n = \vec{c}, \quad i = 1, \ldots, n.
\]
The bounded stable derived category of $R$ introduced by Buchweitz [6] is the quotient category

$$D^b_{\text{sing}}(\text{gr } R) = D^b(\text{gr } R) / D_{\text{perf}}^{\text{gr }}(R)$$

of the bounded derived category $D^b(\text{gr } R)$ of finitely-generated $L$-graded $R$-modules by its full subcategory $D_{\text{perf}}^{\text{gr }}(R)$ consisting of bounded complexes of projectives. This category originates from the theory of matrix factorizations introduced by Eisenbud [8], and studied by Orlov [19] under the name ‘triangulated category of singularities’. This category is not necessarily closed under direct summands, and its idempotent completion will be denoted by $D_{\text{sing}}^{\pi}(\text{gr } R)$.

The transpose of the invertible polynomial $W$ is defined by

$$W^* = \sum_{i=1}^{n} x_1^{a_{1i}} \cdots x_n^{a_{ni}},$$

which can be perturbed to an exact symplectic Lefschetz fibration with respect to the standard Euclidean Kähler form on $\mathbb{C}^n$. Let $\mathcal{F}^b W^*$ be the directed $A_{\infty}$-category defined by Seidel [26, 27] whose set of objects is a distinguished basis of vanishing cycles and whose spaces of morphisms are Lagrangian intersection Floer complexes.

The following conjecture comes from the combination of transposition mirror symmetry by Berglund and Hübsch [3] and homological mirror symmetry by Kontsevich [14]:

**Conjecture 1.1** For an invertible polynomial $W$, there is an equivalence

$$D^b_{\text{sing}}(\text{gr } R) \cong D^b \mathcal{F}^b W^*$$

(1.1)

of triangulated categories.

Conjecture 1.1 is known to hold for Brieskorn–Pham singularities [11]. Takahashi and Ebeling [9, 29] studies Conjecture 1.1 from the point of view of the duality of regular systems of weights by Saito [23].

Recall that polynomials of types $A_n$ and $D_n$ are defined by $x^n+1$ and $x^{n-1} + xy^2$ respectively. We prove the following in this paper:

**Theorem 1.2** One has an equivalence

$$D_{\text{sing}}^{\pi}(\text{gr } R) \cong D^b \mathcal{F}^b W^*$$

(1.2)

of triangulated categories if $W^*$ is a Sebastiani–Thom sum of polynomials of types $A$ or $D$.

The proof is based on the study of the behavior of categories on both sides of (1.2) under the Sebastiani–Thom summation of a polynomial of type $D$.

The organization of this paper is as follows: In Sect. 2, we describe the Fukaya category of a Lefschetz fibration defined by a polynomial of type $D$ in terms of a quiver of the same type. Such a description is first given by Seidel [25, Proposition 3.4], and our exposition is organized in such a way that is convenient for the inductive description of the Fukaya category of a Sebastiani–Thom sum of polynomials of types $A$ and $D$ in Sect. 3. The bounded stable derived category of the transpose of a type $D$ singularity is computed in Sect. 4, and the behavior of stable derived categories under Sebastiani–Thom summation of polynomials is studied in Sect. 5. In Sect. 6, we discuss a possible generalization of Conjecture 1.1 to the case with group actions when $n = 2$. 

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