Certain classes of pluricomplex Green functions
don $\mathbb{C}^n$

Dan Coman

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-5683, USA
(e-mail: Dan.F.Coman.2@nd.edu)

Received November 24, 1998; in final form April 19, 1999 /
Published online July 3, 2000 – © Springer-Verlag 2000

Abstract. We consider (pluricomplex) Green functions defined on $\mathbb{C}^n$, with logarithmic poles in a finite set and with logarithmic growth at infinity. For certain sets, we describe all the corresponding Green functions. The set of these functions is large and it carries a certain algebraic structure. We also show that for some sets no such Green functions exist. Our results indicate the fact that the set of poles should have certain algebro-geometric properties in order for these Green functions to exist.

Mathematics Subject Classification (1991): 32F05, 32F07, 31C10

1 Introduction

For a bounded domain $D \subset \mathbb{C}^n$, let $A = \{p_1, \ldots, p_k\}$ be a $k$-tuple of pairwise distinct points in $D$ and let $W = \{\nu_1, \ldots, \nu_k\}$ be a $k$-tuple of positive real numbers. The pluricomplex Green function of $D$ with poles in $A$ of weights in $W$, is defined by $g_D(z, A, W) = \sup u(z)$, where the supremum is taken over the class of negative plurisubharmonic functions $u$ in $D$ which have a logarithmic pole at $p_j$ with weight $\nu_j$, for $j = 1, \ldots, k$.

These functions exist for any $k$-tuples $A, W$, and their definition is in analogy to the one dimensional case (see [Kl1], [L]). In [D] and [L] it is shown that if $\Omega$ is hyperconvex then $g_D(\cdot, A, W)$ is the unique solution to the following Dirichlet problem for the complex Monge-Ampère operator: $u \in PSH(D) \cap C(\overline{D}\setminus A)$, $u(z) - \nu_j \log \|z - p_j\| = O(1)$ as $z \to p_j$, $(dd^c u)^n = \sum_{j=1}^k \nu_j \delta_{p_j}$, $u = 0$ on $\partial D$. Here, as well as in the sequel, $PSH(D)$ denotes the class of plurisubharmonic functions on $D$, $d = \partial + \overline{\partial}$, $d^c = \frac{1}{2\pi i} (\partial - \overline{\partial})$, and $\delta_{p_j}$ is the Dirac mass at $p_j$. 
In the case of entire plurisubharmonic functions, \( L_+ \) is defined to be the set of functions \( u \in PSH \cap L^{\infty}_{loc}(\mathbb{C}^n) \) such that \( u(z) = \log \|z\| + O(1) \) as \( \|z\| \to \infty \) (see e.g. [B] and [Kl2]). The following Monge-Ampère equation is studied in [Ko1] and [Ko2]: \( u \in L_+ \), \( (dd^c u)^n = f \, d\lambda \), where \( \lambda \) is the Lebesgue measure and \( f \) is a non-negative measurable function. In [Ko1] it is shown that this equation has a continuous solution for certain classes of functions \( f \), and regularity questions are studied in [Ko2]. The uniqueness of the solution is proved in [BT3]: if \( u,v \in L_+ \) and \( (dd^c u)^n = (dd^c v)^n \) then \( u - v \) is constant.

In this paper we study entire plurisubharmonic functions with logarithmic growth (in analogy to the class \( L_+ \)), whose Monge-Ampère measure is a finite sum of Dirac masses. Let \( A = \{ p_1, \ldots, p_k \} \) be a \( k \)-tuple of pairwise distinct points in \( \mathbb{C}^n \) and \( W = \{ \nu_1, \ldots, \nu_k \} \) be a \( k \)-tuple of positive weights. Without loss of generality we shall assume throughout the paper that \( \nu_1 \geq \nu_2 \geq \ldots \geq \nu_k > 0 \). As in the case of a bounded domain, we can consider the following classes \( M_n(A,W) \) of entire pluricomplex Green functions. By definition, \( M_n(A,W) \) is the set of solutions of the following Monge-Ampère equation:

\[
\begin{cases}
  u \in PSH(\mathbb{C}^n) \cap L^{\infty}_{loc}(\mathbb{C}^n \setminus A) , \\
  u(z) - \nu_j \log \|z - p_j\| = O(1) \quad \text{as } z \to p_j , \\
  \lim_{\|z\| \to \infty} \left( \frac{u(z)}{\log \|z\|} \right) \in (0, +\infty) \quad \text{exists} , \\
  (dd^c u)^n = \sum_{j=1}^k \nu_j^n \delta_{p_j} .
\end{cases}
\]

When the weights \( \nu_j \) are equal, hence without loss of generality equal to 1, we denote the corresponding class by \( M_n(A) \). If \( A \) consists of one point, we may assume \( A = \{0\} \), \( \nu_1 = 1 \), and we denote the corresponding class by \( M_n(0) \).

We remark that \( M_1(A,W) \) consists precisely of the functions \( \sum_{j=1}^k \nu_j \log \|z - p_j\| + c \), for arbitrary constants \( c \). Functions in \( M_n(A,W) \) can be viewed as higher dimensional generalizations of these classical Green functions. However, we shall see that when \( n > 1 \) \( M_n(A,W) \) can be in some cases empty. In some other cases we will describe all the elements of \( M_n(A,W) \). In these cases \( M_n(A) \) has actually many elements, in the sense that they are not unique up to addition of constants.

In Sect. 2 of the paper we collect a few known results. In Sect. 3 we study the case when \( A = f^{-1}(0) \), where \( f : \mathbb{C}^n \to \mathbb{C}^n \) is a holomorphic map such that 0 is a regular value for \( f \) and the following holds for some integer