New Cartesian grid methods for interface problems using the finite element formulation

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Received February 18, 1999 / Revised version received October 7, 2002 / Published online June 6, 2003 – © Springer-Verlag 2003

Summary. New finite element methods based on Cartesian triangulations are presented for two dimensional elliptic interface problems involving discontinuities in the coefficients. The triangulations in these methods do not need to fit the interfaces. The basis functions in these methods are constructed to satisfy the interface jump conditions either exactly or approximately. Both non-conforming and conforming finite element spaces are considered. Corresponding interpolation functions are proved to be second order accurate in the maximum norm. The conforming finite element method has been shown to be convergent. With Cartesian triangulations, these new methods can be used as finite difference methods. Numerical examples are provided to support the methods and the theoretical analysis.

Mathematics Subject Classification (2000): 65L10, 65L60, 65L70

1 Introduction

In this paper, we develop finite-element immersed interface methods using Cartesian grids for differential equations with discontinuities in the coefficients across one or several arbitrary interfaces in the solution domain. These problems are referred to as interface problems in this paper. A model problem

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is
\[-\nabla \cdot (\beta \nabla u) = f, \quad (x, y) \in \Omega,\]
\[u|_{\partial \Omega} = 0,\]
(1.1)
defined in a domain \(\Omega\) with an immersed interface \(\Gamma\), see Fig. 1.1 for an illustration. A vast collection of applications involve solving such an equation, for example, the projection method for solving Navier-Stokes equations involving two phase flow [4, 8, 21, 44], the Hele-Shaw flow [19, 20] and many others. If a problem of interest involving two different materials, such as water and air, solid and liquid in solidification problems, the coefficient \(\beta\) will typically have a jump across the interface between two materials. In some cases, the jump can be very big, for example, the ratio of the density of the air and water is about 1:1000 in the magnitude.

Our methods can also be applied to those models whose source term \(f\) in (1.1) have a delta function singularity, for example
\[f(x) = f_c(x) - \int_{\Gamma} Q(X(s)) \delta(x - X(s)) \, ds\]
where \(f_c\) is a bounded function, \(\delta\) is the two dimensional Dirac-delta function, \(X(s)\) is the arc-length parameterization of the interface \(\Gamma\), and \(Q(X(s))\) is the source strength on the interface. The expression above can also be written as
\[f = f_c - Q \delta_{\Gamma}.\]
(1.2)
Such a source function is one of the most important features of Peskin’s immersed boundary method (IBM) [40, 41], which has been used for many problems in mathematical biology and computational fluid mechanics, see for example, [5, 13, 14, 16, 43] and many others.

When \(\beta \in C^2\) in \(\Omega^- \cup \Omega^+\) excluding the interface \(\Gamma\), see Fig.1.1, then \(u(x, y) \in H^1\), see [6]. From equation (1.1) and (1.2), it is easy to obtained the jump conditions
\[\left[ u \right]_\Gamma = u(x, y)^+ - u(x, y)^- = 0, \quad \text{continuity condition},\]
\[\left[ \beta \frac{\partial u}{\partial n} \right]_\Gamma = \beta^+ \frac{\partial u^+}{\partial n} - \beta^- \frac{\partial u^-}{\partial n} = Q(s), \quad \text{net flux across the interface},\]
(1.4)
where the jump is defined as the difference of the limiting values from the outside of the interface to the inside, and \(\frac{\partial u^+}{\partial n}\) is the normal derivative of the solution. Therefore, the model interface problem can be written in an equivalent form:
\[-\nabla \cdot (\beta \nabla u) = f_c, \quad (x, y) \in \Omega - \Gamma, \quad f_c \in L^2(\Omega),\]
\[\left[ u \right]_\Gamma = 0, \quad \left[ \beta \frac{\partial u}{\partial n} \right]_\Gamma = Q(s), \quad u|_{\partial \Omega} = 0.\]
(1.5)