Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients

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Summary. General error estimates are proved for a class of finite element schemes for nonstationary thermal convection problems with temperature-dependent coefficients. These variable coefficients turn the diffusion and the buoyancy terms to be nonlinear, which increases the nonlinearity of the problems. An argument based on the energy method leads to optimal error estimates for the velocity and the temperature without any stability conditions. Error estimates are also provided for schemes modified by approximate coefficients, which are used conveniently in practical computations.

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1 Introduction

We are concerned here with finite element methods for thermal convection problems with temperature-dependent coefficients. In several cases of constant coefficients, error estimates of finite element approximations have already been developed. Some stationary problems have been studied by Bernardi et al. [1] and Boland and Layton [3]; a nonstationary one in a semi-discrete form by Boland and Layton [2]; some simplified nonstationary ones with infinite Prandtl number by Itoh and Tabata [12], Tabata and Suzuki [17], and Tagami and Itoh [20]. On the other hand, as pointed out by Getling [6], we often need to take into account the variation of material coefficients depending
on the temperature in physical and engineering problems. Especially, variable coefficients play an important role in the formation of convection patterns. Our aim of this paper is to establish general error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients.

As far as we know, there are few researches on full-discrete finite element methods for nonstationary thermal convection problems with variable coefficients, and even with constant coefficients, from the mathematical point of view. For a simplified thermal convection problem error estimates have been established in the case of temperature-dependent coefficients by Tabata [15] and Tabata and Suzuki [18]. Their mathematical model does not include the inertia terms in the motion of fluid, because the slow velocity case is considered and the Prandtl number is set to be infinity. In this paper the original thermal convection problem without such reduction is treated. More precisely, we consider the following nonstationary thermal convection problem with temperature-dependent coefficients: Find the velocity \( u \), the pressure \( p \), and the temperature \( \theta \):

\[
\begin{align*}
(u, p, \theta) : & \quad \Omega \times (0, T) \rightarrow \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \\
\text{such that} & \\
(1a) & \quad \partial_t u + (u \cdot \nabla) u - \nabla \cdot [\nu(\theta) \nabla u] + \nabla p - \beta(\theta) \theta = f \quad \text{in } \Omega \times (0, T), \\
(1b) & \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T), \\
(1c) & \quad \partial_t \theta + (u \cdot \nabla) \theta - \nabla \cdot (\kappa(\theta) \nabla \theta) = g \quad \text{in } \Omega \times (0, T), \\
(1d) & \quad u = u_D \quad \text{on } \Gamma \times (0, T), \\
(1e) & \quad \theta = \theta_D \quad \text{on } \Gamma \times (0, T), \\
(1f) & \quad u = u^0 \quad \text{in } \Omega \text{ at } t = 0, \\
(1g) & \quad \theta = \theta^0 \quad \text{in } \Omega \text{ at } t = 0,
\end{align*}
\]

where \( T > 0 \) denotes a time; \( \Omega \) a bounded domain in \( \mathbb{R}^d \) \((d = 2, 3)\) with Lipschitz-continuous boundary \( \Gamma \);

\[
(f, g) : \quad \Omega \times (0, T) \rightarrow \mathbb{R}^d \times \mathbb{R}
\]

a set of external force and heat source;

\[
(u_D, \theta_D) : \quad \Gamma \times (0, T) \rightarrow \mathbb{R}^d \times \mathbb{R}
\]

a set of boundary velocity and temperature;

\[
(u^0, \theta^0) : \quad \Omega \rightarrow \mathbb{R}^d \times \mathbb{R}
\]