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Error estimates for a mixed finite volume method for the $p$-Laplacian problem

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Abstract In this work we propose and analyze a mixed finite volume method for the $p$-Laplacian problem which is based on the lowest order Raviart–Thomas element for the vector variable and the $P_1$ nonconforming element for the scalar variable. It is shown that this method can be reduced to a $P_1$ nonconforming finite element method for the scalar variable only. One can then recover the vector approximation from the computed scalar approximation in a virtually cost-free manner. Optimal a priori error estimates are proved for both approximations by the quasi-norm techniques. We also derive an implicit error estimator of Bank–Weiser type which is based on the local Neumann problems.

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1 Introduction

In this paper we consider the following nonlinear $p$-Laplacian problem with the Dirichlet boundary condition

$$
\begin{cases}
-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega$ is an open bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial \Omega$, $f \in L^p(\Omega)$, and $g \in W^{1-1/p, p}(\partial \Omega)$, with $p'$ being the conjugate exponent of $p \in (1, \infty)$, i.e., $1/p + 1/p' = 1$. This problem is one of the typical examples of

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degenerate nonlinear systems arising from, e.g., power-law materials and quasi-Newtonian flows.

The weak formulation for problem (1.1) reads as follows:

\[
\text{find } u \in W^{1,p}_g(\Omega) := \{ v \in W^{1,p}(\Omega) : v|_{\partial\Omega} = g \} \text{ such that }
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} fv \, dx \quad \forall v \in W^{1,p}_0(\Omega).
\] (1.2)

This is equivalent to the minimization problem

\[
J(u) \leq J(v) \quad \forall v \in W^{1,p}_g(\Omega),
\] (1.3)

where

\[
J(v) = \frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \int_{\Omega} fv \, dx.
\] (1.4)

It is well known (cf. [7,8]) that problem (1.2) is well posed.

Finite element approximations for problem (1.2) have been extensively studied by many authors; see, for example, [7,8,12,19,21] for some previous works on a priori and a posteriori error estimation in the conventional $W^{1,p}(\Omega)$-norm. Sharper a priori error estimates were derived in [3,4,14] by developing the quasi-norm techniques. In [15–17] these techniques were extended to establish improved a posteriori error estimators of residual type for the $P1$ conforming and nonconforming finite element methods.

In some applications it is of primary interest to gain accurate approximations for the vector $\sigma = |\nabla u|^{p-2} \nabla u$. For this purpose, one rewrites problem (1.1) in the mixed form

\[
\begin{cases}
\sigma - |\nabla u|^{p-2} \nabla u = 0 & \text{in } \Omega, \\
\nabla \cdot \sigma + f = 0 & \text{in } \Omega, \\
u = g & \text{on } \partial\Omega.
\end{cases}
\] (1.5)

The mixed finite element method for this system was studied by Farhloul [11] in the case of $1 < p < 2$.

The goal of this paper is to propose and analyze a mixed finite volume method for the system (1.5) using the quasi-norm techniques of [3,15–17]. This method was first introduced for Poisson’s equation (namely, $p = 2$) by Courbet and Croisille [9] who called it the finite volume box method. They considered the lowest order Raviart–Thomas space for the vector variable and the $P1$ nonconforming space for the scalar variable, and discretized the mixed system (1.5) by integrating the equations over each element of a given triangulation, based on the notion of the box method. This method was later extended to more general situations, such as tensor coefficients and quadrilateral grids, by Chou et al. [6].

The main advantages of the above mixed finite volume method can be summarized as follows (cf. [6]):

- Mass is conserved locally on each element.
- Even though the vector variable has continuous normal components, it is possible to decouple it locally from the scalar variable, without using Lagrange multipliers.