A mixed finite volume scheme for anisotropic diffusion problems on any grid

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Abstract We present a new finite volume scheme for anisotropic heterogeneous diffusion problems on unstructured irregular grids, which simultaneously gives an approximation of the solution and of its gradient. The approximate solution is shown to converge to the continuous one as the size of the mesh tends to 0, and an error estimate is given. An easy implementation method is then proposed, and the efficiency of the scheme is shown on various types of grids and for various diffusion matrices.

Keywords Finite volume scheme · Unstructured grids · Irregular grids · Anisotropic heterogeneous diffusion problems

1 Introduction

The computation of an approximate solution for equations involving a second order elliptic operator is needed in so many physical and engineering areas, where the efficiency of some discretization methods, such as finite difference, finite element or finite volume methods, has been proved. The use of finite volume methods is particularly popular in the oil engineering field, since it allows...
for coupled physical phenomena in the same grids, for which the conservation of various extensive quantities appears to be a main feature. However, it is more challenging to define convergent finite volume schemes for second-order elliptic operators on refined, distorted or irregular grids, designed for the purpose of another problem.

For example, in the framework of geological basin simulation, the grids are initially fitted on the geological layers boundaries, which is a first reason for the loss of orthogonality. Then, these grids are modified during the simulation, following the compaction of these layers (see [16]), thus leading to irregular grids, as those proposed by [18]. As a consequence, it is no longer possible to compute the fluxes resulting from a finite volume scheme for a second order operator by a simple two-point difference across each interface between two neighboring control volumes. Such a two-point scheme is consistent only in the case of an isotropic operator, using a grid such that the lines connecting the centers of the control volumes are orthogonal to the edges of the mesh.

The problem of finding a consistent expression using only a small number of points, for the finite volume fluxes in the general case of any grid and any anisotropic second order operator, has led to many works (see [1–3,16] and references therein; see also [22]). A recent finite volume scheme has been proposed [13,14], permitting to obtain a convergence property in the case of an anisotropic heterogeneous diffusion problem on unstructured grids, which all the same satisfy the above orthogonality condition. In the case where such an orthogonality condition is not satisfied, a classical method is the mixed finite element method which also gives an approximation of the fluxes and of the gradient of the unknown (see [4–6,24] for example, among a very large literature).

Note that, although the Raviart–Thomas basis is not directly available on control volumes which are not simplices or regular polyhedra, such a basis can be built on more general irregular grids. In [20], such a construction is completed using decomposition into simplices and a local elimination of the unknowns at the internal edges. In [11,15], such basis functions are obtained from the resolution of a Neumann elliptic problem in each grid block. However, it has been observed that the use of mixed finite element method could demand high refined grids on some highly heterogeneous and anisotropic cases (see [21] and the numerical results provided in the present paper). An improvement of the mixed finite element scheme is the expanded mixed finite element scheme [7,8], where different discrete approximations are proposed for the unknown, its gradient and the product of the diffusion matrix by the gradient of the unknown; however, this last scheme seems to present the same restrictions on the meshes as the mixed finite element scheme. Klausen and Russell [19] gives a review of different “mixed” methods, albeit mostly on structured (or not very general) grids.

We thus propose in this paper an original finite volume method, called the mixed finite volume method, which can be applied on any type of grids in any space dimension, with very few restrictions on the control volumes. The implementation of this scheme is proved to be easy, and no geometric complex shape function has to be computed. Accurate results are obtained on coarse irregular