R. Hiptmair · G. Widmer · J. Zou

Auxiliary space preconditioning in $H_0(\text{curl}; \Omega)$

Abstract We adapt the principle of auxiliary space preconditioning as presented in [J. Xu, The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids, Computing, 56 (1996), pp. 215–235.] to $H(\text{curl}; \Omega)$-elliptic variational problems discretized by means of edge elements. The focus is on theoretical analysis within the abstract framework of subspace correction. Employing a Helmholtz-type splitting of edge element vector fields we can establish asymptotic $h$-uniform optimality of the preconditioner defined by our auxiliary space method.

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1 Introduction

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, we are concerned with solving the boundary value problem

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R. Hiptmair
SAM, ETZ Zürich, 8092 Zürich,
E-mail: hiptmair@sam.math.ethz.ch

G. Widmer
SAM, ETH Zürich, 8092 Zürich,
E-mail: gisela.widmer@sam.math.ethz.ch

J. Zou
Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong
E-mail: zou@math.cuhk.edu.hk
\[
\text{curl} \ \text{curl} \ u + u = f \quad \text{in } \Omega, \quad u_t = 0 \quad \text{on } \partial \Omega, \tag{1.1}
\]

where \( f \in (L^2(\Omega))^3 \) and \( u_t \) stands for the tangential trace of the vector field \( u \). Of course, (1.1) could be supplemented by coefficients with spatial variation and these will usually occur when (1.1) is used to model, e.g., low frequency electromagnetic fields [4]. However, in order to keep the presentation focused, we forgo the treatment of variable coefficients. This generalization will be discussed in a few remarks.

The weak formulation of (1.1) is straightforward and reads: seek \( u \in H_0(\text{Curl}; \Omega) \) such that

\[
a(u, v) := \int_\Omega \text{curl} \ u \cdot \text{curl} \ v + u \cdot v \, dx = \int_\Omega f \cdot v \, dx \quad \forall v \in H_0(\text{Curl}; \Omega). \tag{1.2}
\]

For the definition and properties of the function spaces used in this paper we refer to [7, Ch. 1]. In fact, \( a(\cdot, \cdot) \) agrees with the natural inner product of the Hilbert space \( H_0(\text{Curl}; \Omega) \). This ensures existence and uniqueness of solutions of (1.2).

The large sparse symmetric positive definite linear system of equations arising from a finite element Galerkin discretization of (1.2) calls for efficient preconditioning. The search for preconditioners can be guided by the insight that (1.1) is close kin to

\[
-\Delta u + u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Omega. \tag{1.3}
\]

In fact, both boundary value problems arise from a single formula when stated in the calculus of differential forms [13]. This suggests that established concepts for preconditioning in the case of second order elliptic boundary value problems carry over to (1.1), with some adjustments, though, in order to deal with the kernel of the \( \text{curl} \)-operator. This guideline has proved very successful in the case of multigrid methods [1, 9, 12] and various domain decomposition approaches [8, 16–18, 27].

Now we set out to perform the adaptation for another powerful idea developed for (1.3): the idea of auxiliary space preconditioning [5, 6, 29]. Taking the cue from [12], we supplement the raw principle of auxiliary space preconditioning with a special treatment of \( \text{curl} \)-free functions. This gives rise to the algorithm discussed in Section 4. A key tool in its theoretical analysis will be a regular \textit{Helmholtz-type decomposition} of \( H_0(\text{curl}; \Omega) \) [14, 15, 21]. Details will be given in Section 5. The final result is that the new auxiliary space method for (1.1) actually provides an asymptotically optimal preconditioner, whose performance does not degrade as the resolution of the mesh is increased.

The relevance of auxiliary space preconditioning is due to the fact that it targets linear systems of equations that emerge from finite element discretization on a big unstructured mesh for which no refinement hierarchy is available. This is exactly the class of problems for which algebraic multigrid (AMG) has been conceived [23, 26]. Very efficient AMG algorithms are available for (1.3), but AMG approaches to (1.1) fail to deliver the usual mesh-independent multigrid efficiency [2, 22]. This can only be achieved by geometric multigrid algorithms that rely on nested meshes [1, 12, 25].

The auxiliary space method manages to harness the power of standard multigrid by employing auxiliary meshes that, unlike the original unstructured mesh,