Finite volume transport schemes

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1 Introduction

We address the numerical analysis in $L^1$, $L^2$ and $L^\infty$ of the high order finite volume schemes recently derived in [2,11,12] for discretization of the advection equation

$$\partial_t u + a \partial_x u = 0.$$ 

We shall assume $a > 0$ for simplicity. These schemes may written in conservation form on an uniform mesh
\[
\frac{u^n_j - u^n_j}{\Delta t} + a \frac{u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}}}{\Delta x} = 0. \tag{1}
\]

It is sufficient to define the flux \(u^n_{j+\frac{1}{2}}\) in function of the neighboring values to completely define the scheme. The Courant number is

\[\nu = \frac{a}{\Delta x}.\]

Some low and high order finite volume transport schemes are shown in Table 2. In our context a finite volume scheme is based on some generalization of the upwind scheme in the spirit of the Lax–Wendroff scheme.

Obtaining optimal results of convergence in \(L^d\) is a consequence of their stability in \(L^d\). By definition

\[||v||_{L^d} = \left( \int_{\mathbb{R}} |v(x)|^d dx \right)^{\frac{1}{d}} = (\Delta x \sum |v_j|^d)^{\frac{1}{d}}, \quad 1 \leq d < \infty,
\]

and \(||v||_{L^\infty} = \sup |v(x)| = \sup |v_j|\). In practice \(d = 1, 2\) or \(\infty\) are the most interesting cases. A classical theorem of Godunov states first order linear schemes are the only linear ones that satisfy the maximum principle. Asymptotic stability in \(L^1\) may be useful to evaluate the oscillating behavior of high order schemes.

**Definition 1** (A-stability) We say a scheme is A-stable (asymptotic stability) in \(L^d\) if there exists a bound \(K > 0\) which does not depend on \(\Delta x\), \(\nu\) such that

\[||u^n||_{L^d} \leq K ||u^0||_{L^d}, \quad \forall n.
\]

A-stability is more stringent than the usual uniform stability of Lax [7,10] or Godunov [4], for which the constant \(K\) may depend on some final time \(T > 0\) and the estimate is valid for \(n\Delta t \leq T\). For advection the requirement of A-stability is nevertheless very natural since it is a property of the exact solution.

As usual the analysis in \(L^2\) is based on Fourier analysis, see [5]. All standard advection schemes are A-stable in \(L^2\) (with \(K = 1\)) and therefore have optimal convergence properties in \(L^2\) even for discontinuous initial data. For example we will prove the following result. Consider an initial data in \(L^\infty \cap BV\) function [8]

\[|u_0|_{BV} = \lim sup_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{|u_0(x) - u_0(x - \varepsilon)|}{\varepsilon} dx.
\]

If \(u_0\) is differentiable in \(L^1\) then \(|u_0|_{BV} = ||u'_0||_{L^1}\). In the appendix we show the order of convergence in \(L^2\) of the numerical solution to the exact solution is

\[||u^n - v^n||_{L^2} \leq \left( C ||u_0||_{L^\infty}^\frac{1}{2} |u_0|_{BV}^\frac{1}{2} \right) \times \left( \Delta x^a T^b + \Delta x^\frac{1}{2} \right)
\]

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