Analysis of a BDF–DGFE scheme for nonlinear convection–diffusion problems

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Abstract We deal with the numerical solution of a scalar nonstationary nonlinear convection–diffusion equation. We employ a combination of the discontinuous Galerkin finite element method for the space semi-discretization and the $k$-step backward difference formula for the time discretization. The diffusive and stabilization terms are treated implicitly whereas the nonlinear convective term is treated by a higher order explicit extrapolation method, which leads to the necessity to solve only a linear algebraic problem at each time step. We analyse this scheme and derive a priori asymptotic error estimates in the discrete $L^\infty(L^2)$-norm and the $L^2(H^1)$-seminorm with respect to the mesh size $h$ and time step $\tau$ for $k = 2, 3$. Numerical examples verifying the theoretical results are presented.

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1 Introduction

Our aim is to develop a sufficiently efficient, robust and accurate numerical scheme for the simulation of unsteady viscous compressible flows. During the last years, the...
so-called discontinuous Galerkin method (DGM) seems to be a promising technique for the solution of the compressible Navier–Stokes equations, see e.g. [7–10,22,32–34,38,39,42,51,52]. DGM is based on a piecewise polynomial but discontinuous approximation where the interelement continuity is replaced by additional stabilization terms.

There are several variants of the DGM for the solution of problems containing diffusion terms. It is possible to use primitive variables or a mixed method. The method can be stabilized with the aid of a symmetric or nonsymmetric treatment of diffusion terms, often combined with interior and boundary penalty terms. Among methods using primitive variables, two approaches, SIPG (symmetric interior penalty Galerkin [1]) and NIPG (nonsymmetric interior penalty Galerkin [48]), are very popular. These techniques represent a generalization of the boundary penalty terms proposed by Babuška and Zlámal allowing to impose the Dirichlet boundary condition in a weak sense instead of building it in the finite element space (see [5]). The nonsymmetric variant (NIPG) does not give an optimal order of convergence in the \( L^2 \)-norm for elliptic problems (see [2,47]), but it leads to a coercive operator for an arbitrary positive penalty coefficient.

On the other hand, the symmetric variant (SIPG) gives optimal order of convergence provided that penalty coefficient is chosen sufficiently large. We analysed both approaches in [21] (NIPG) and [19] (SIPG) for a scalar convection–diffusion equation. There is a number of works devoted to theory and applications of the DGM. Let us mention, e.g., [6,10,15,20,25,36,44,47]. For a survey of various discontinuous Galerkin techniques, see, e.g. [2,12,13].

For time-dependent problems, it is possible to use a discontinuous approximation also for the time discretization (e.g. [24,49,51,52]), but the most standard approach is the method of lines when the space semi-discretization leads to a system of ordinary differential equations (ODEs) formally written in the form

\[
\frac{d}{dt} w(t) = F^l(w(t)) + F^n(w(t)), \quad t \in (0, T)
\]

(1)

where \( w : (0, T) \rightarrow \mathbb{R}^m, \ m \gg 1 \) and \( F^l(\cdot) \) and \( F^n(\cdot) \) are linear and nonlinear functions of their arguments, respectively.

The system (1) should be solved by a suitable ODE technique. In this case, the explicit Runge–Kutta schemes are very popular due to their simplicity and high order of accuracy, see [8,10,12,15]. However, the drawback of explicit schemes is a strong restriction on the length of the time step. To avoid this disadvantage it is convenient to use an implicit time discretization, e.g., *backward differential formulae* (BDF), which are, since the works [27,28], widely used for the solution of stiff ODEs, see [30,31].

However, fully implicit schemes lead to a nonlinear system of algebraic equations at each time step, whose solution is rather complicated and expensive. Therefore, the so-called *implicit-explicit methods* (IMEX) are applied to the discretization of (1), where the linear term is discretized implicitly and the nonlinear one explicitly. Then we solve a linear algebraic problem at each time step and the IMEX schemes have better stability properties than explicit ones. Multi-steps IMEX schemes were analysed in [14] for parabolic equations, see also [45] for an analysis of IMEX techniques with