Pathwise approximation of stochastic differential equations on domains: higher order convergence rates without global Lipschitz coefficients

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Abstract We study the approximation of stochastic differential equations on domains. For this, we introduce modified Itô–Taylor schemes, which preserve approximately the boundary domain of the equation under consideration. Assuming the existence of a unique non-exploding solution, we show that the modified Itô–Taylor scheme of order $\gamma$ has pathwise convergence order $\gamma - \varepsilon$ for arbitrary $\varepsilon > 0$ as long as the coefficients of the equation are sufficiently differentiable. In particular, no global Lipschitz conditions for the coefficients and their derivatives are required. This applies for example to the so called square root diffusions.

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1 Introduction

In this article, we study the approximation of the Itô stochastic differential equation

$$dx(t) = f(x(t)) \, dt + g(x(t)) \, dW(t), \quad t \geq 0, \quad x(0) = x_0, \quad (1)$$

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which takes values in a domain \( D \subset \mathbb{R}^d \). Here, \( f : D \to \mathbb{R}^d \), \( g : D \to \mathbb{R}^{d,m} \), \( W = (W(t), t \geq 0) \) is a \( m \)-dimensional Brownian motion adapted to a filtration \( \mathcal{F} = (\mathcal{F}_t, t \geq 0) \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), and \( x_0 \) is a \( \mathcal{F}_0 \)-measurable random variable, which is independent of \( W \).

In many applications, the drift- or diffusion coefficients of Eq. (1) have a simple structure, e.g. polynomial, but fail to satisfy a global Lipschitz condition.

For example, the stochastic Volterra–Lotka system

\[
\begin{align*}
dx(t) &= \text{diag}(x_1(t), \ldots, x_d(t)) \left[ (A + Bx(t)) \, dt + Cx(t) \, dW(t) \right],
\end{align*}
\]  

(2)

where \( A \in \mathbb{R}^d \), \( B \), \( C \in \mathbb{R}^{d,d} \) and \( (W(t), t \geq 0) \) is a one-dimensional Brownian motion, see, e.g. [17], has polynomial coefficients and scalar noise. Another example with polynomial coefficients is the stochastic Duffing–van der Pol equation (see [3])

\[
\begin{align*}
dx_1(t) &= x_2(t) \, dt, \\
dx_2(t) &= (\alpha x_1(t) + \beta x_2(t) - x_1^3(t) - x_1^2(t)x_2(t) - \frac{\sigma_1^2}{2} x_1(t) - \frac{\sigma_2^2}{2} x_2(t)) \, dt \\
&\quad + \sigma_1 x_1(t) \, dW^{(1)}(t) + \sigma_2 x_2(t) \, dW^{(2)}(t) + \sigma_3 \, dW^{(3)}(t),
\end{align*}
\]  

(3)

where \( \alpha, \beta, \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \) and \( W^{(1)}, W^{(2)} \) and \( W^{(3)} \) are three independent scalar Brownian motions.

Moreover, in mathematical biology and financial mathematics, see, e.g. [14,21], particular interest has been given to stochastic differential equations of the so called Cox–Ingersoll–Ross type

\[
\begin{align*}
dx(t) &= \kappa (\lambda - x(t)) \, dt + \theta \sqrt{|x(t)|} \, dW(t),
\end{align*}
\]  

(4)

where \( \kappa, \lambda, \theta \geq 0 \). Here the diffusion coefficient neither satisfies a global Lipschitz condition nor is differentiable at \( x = 0 \). However, if \( \kappa \lambda \geq \theta^2 / 2 \) and \( x_0 \in (0, \infty) \), then the boundary zero is unattainable. Hence in this case the solution of Eq. (4) never leaves the set \( D = (0, \infty) \) and the coefficients are infinitely differentiable on \( D \). For further examples, we refer to [2,3,14,17,21].

In this article, we will consider the approximation of Eq. (1) with respect to the pathwise maximum error in the discretization points

\[
\sup_{i=1,\ldots,n} |x(t_i, \omega) - \bar{x}(t_i, \omega)|, \quad \omega \in \Omega,
\]  

(5)

where \( t_1, \ldots, t_n \) are the nodes of the discretization. This error criterion is in particular appropriate for equations with a non-integrable initial value and for stochastic dynamics, since the theory of random dynamical systems is of pathwise nature. Approximation methods for stochastic differential equations with respect to pathwise error criteria have been considered, e.g. in [6,8,9,16,24]. While in [16,24] pathwise convergence rates for several approximation schemes (including the standard Itô–Taylor schemes)