Modified Lagrange–Galerkin methods of first and second order in time for convection–diffusion problems

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Abstract We introduce modified Lagrange–Galerkin (MLG) methods of order one and two with respect to time to integrate convection–diffusion equations. As numerical tests show, the new methods are more efficient, but maintaining the same order of convergence, than the conventional Lagrange–Galerkin (LG) methods when they are used with either $P_1$ or $P_2$ finite elements. The error analysis reveals that: (1) when the problem is diffusion dominated the convergence of the modified LG methods is of the form $O(h^{m+1} + h^2 + \Delta t^q)$, $q = 1$ or 2 and $m$ being the degree of the polynomials of the finite elements; (2) when the problem is convection dominated and the time step $\Delta t$ is large enough the convergence is of the form $O(h^{m+1} + h^2 + \Delta t^q)$; (3) as in case (2) but with $\Delta t$ small, then the order of convergence is now $O(h^m + h^2 + \Delta t^q)$; (4) when the problem is convection dominated the convergence is uniform with respect to the diffusion parameter $v(x, t)$, so that when $v \to 0$ and the forcing term is also equal to zero the error tends to that of the pure convection problem. Our error analysis shows that the conventional LG methods exhibit the same error behavior as the MLG methods but without the term $h^2$. Numerical experiments support these theoretical results.

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1 Introduction

We introduce a numerical method that is a modification of the Lagrange–Galerkin method to calculate the numerical solution of convection–diffusion equations. Specifically, we consider the model problem

\[
\begin{align*}
\frac{\partial c}{\partial t} + u \cdot \nabla c &= \nabla \cdot (v(x, t) \nabla c) + f(x, t) \quad \text{in } D \times (0, T), \\
c(x, 0) &= 0, \quad (x, t) \in \partial D \times (0, T), \\
c(x, 0) &= c^0(x), \quad x \in D,
\end{align*}
\]

(1)

where \( D \) is an open bounded domain of \( \mathbb{R}^d(d = 2 \text{ or } 3) \) with a Lipschitz continuous boundary \( \partial D \), \( c : \overline{D} \times [0, T) \rightarrow \mathbb{R}, \ u : D \times (0, T) \rightarrow \mathbb{R}^d \) is a vector-valued function that represents a flow velocity and \( v(x, t) \) is a symmetric positive definite matrix of diffusion coefficients such that for all \( t \), the ratio \( \kappa = \frac{\lambda_{\max}}{\lambda_{\min}} \) is moderate, with \( \lambda_{\max} \) and \( \lambda_{\min} \) denoting the largest and smallest eigenvalues of \( v(x, t) \) respectively. Each component of \( u(x, t) \) is in \( L^\infty(D \times (0, T)) \), \( |u(x, t)| \gg \lambda_{\min} \), and for simplicity we shall consider that \( u(x, t) = 0 \) on \( (x, t) \in \partial D \times t \). If in addition, we assume that the coefficients \( v_{ij}(x, t) \) of the matrix \( v(x, t) \) are in \( L^\infty(D \times (0, T)) \), \( f \in L^2(D \times (0, T)) \), and \( c^0 \in L^2(D) \), then it can be shown that (1) has a unique weak solution \( c \in L^2(0, T; H^1_0(D)) \cap C([0, T]; L^2(D)), \ \frac{\partial c}{\partial t} \in L^2(0, T; H^{-1}(D)) \) that satisfies for each \( v \in H^1_0(D) \) a.e. time \( 0 \leq t \leq T \),

\[
\begin{align*}
\left( \frac{Dc}{Dt}, v \right) + a(t; c, v) &= (f, v), \\
c(x, 0) &= c^0(x),
\end{align*}
\]

(2)

where \( \frac{D}{Dt} := \frac{\partial}{\partial t} + u \cdot \nabla \) is the so-called total derivative, \( a(t; \cdot, \cdot) : H^1_0(D) \times H^1_0(D) \rightarrow \mathbb{R} \) is a continuous coercive bilinear form defined as

\[
a(t; c, v) = \int_D v(x, t) \nabla c \cdot \nabla v dx,
\]

(3)

\( \langle \cdot, \cdot \rangle \) denotes the duality pairing for \( H^1_0 \) and its dual \( H^{-1} \), and \( (\cdot, \cdot) \) is the usual inner product in \( L^2(D) \). To calculate numerically the weak solution by the Lagrange–Galerkin method proposed in [6] and [9], the interval \( [0, T] \) is divided into subintervals \( [t_{n-1}, t_n], n = 1, 2, \ldots, N, \) of length \( \Delta t_n = t_n - t_{n-1} \) such that \( 0 = t_0 < t_1 < \cdots < t_N = T \), and the total derivative \( \frac{Dc}{Dt} \) is discretized along the characteristic curves.