A Gautschi-type method for oscillatory second-order differential equations

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Summary. We study a numerical method for second-order differential equations in which high-frequency oscillations are generated by a linear part. For example, semilinear wave equations are of this type. The numerical scheme is based on the requirement that it solves linear problems with constant inhomogeneity exactly. We prove that the method admits second-order error bounds which are independent of the product of the step size with the frequencies. Our analysis also provides new insight into the mollified impulse method of García-Archilla, Sanz-Serna, and Skeel. We include results of numerical experiments with the sine-Gordon equation.

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1. Introduction

In this paper we study a numerical method for the solution of systems of second-order differential equations

\[ y'' = -Ay + g(y), \quad y(0) = y_0, \quad y'(0) = y'_0, \]

where \( A \) is a symmetric and positive semi-definite real matrix of arbitrarily large norm. We are interested in using step sizes that are not restricted by the frequencies of \( A \), neither for stability nor for accuracy.

García-Archilla, Sanz-Serna and Skeel [3] recently proposed and analyzed a method for oscillatory differential equations, which they called the mollified impulse method. They obtained error bounds for numerical solutions of (1) which do not deteriorate when the product of the step size with
the frequencies becomes large or, what is potentially worse, is close to multiples of $\pi$. Their method is based on the splitting $u'' = -Au, v'' = g(v)$.

Here we study a method which is instead based on the requirement that it reduces to an exact solver for linear equations (1) with constant inhomogeneity $g$. Such a method, which is simple to construct, can be traced back to an old paper of Gautschi [5]. More recently, in [7] we found methods of this type numerically promising in combination with Krylov subspace techniques for approximating the product of the matrix exponential, or related matrix functions, with a vector. Our positive numerical experience called for a rigorous error analysis of such methods.

The error analysis developed here gives very detailed information about the structure of the error. The error is of second order uniformly in the frequencies. It turns out to be largely determined by a scalar function of two variables which accounts for the mixing of frequencies by the numerical method. As a practical consequence, this can be used for the construction of a suitable filter function which appears in the scheme. Our error and stability analysis provides also new insight for the mollified impulse method.

The methods considered in this paper require, in every time step, the computation of the product $\varphi(h^2A)v$ of analytic functions $\varphi$ of the matrix $A$ scaled by the square of the step size $h$, with a vector. This is easy if the eigen-decomposition of $A$ is available, most notably in pseudospectral methods for nonlinear wave equations. Otherwise (or possibly in combination with a partial eigendecomposition), such matrix-function vector products can be computed with Krylov subspace methods [2, 6]. A further alternative, which appears however less favourable in the present context, is to solve in every time step a linear initial value problem, which is associated with the matrix function in question, by a standard numerical integrator with smaller step sizes.

The paper is organized as follows: In Sect. 2 we present the numerical method and some of its variants, and an extension to more general equations $y'' = f(y) + g(y)$. Section 3 develops the error analysis for Eq. (1), with the main result stated in Theorem 1. A major technical difficulty in this paper is to bound the Schur multiplier norm of matrices composed of values of the error function. Such bounds are derived in Sect. 4. They depart from optimality only by logarithmic terms. Section 5 deals with the fixed-step-size stability of the method for linear problems (1) with $g(y) = -By$ for positive semi-definite $B$. Section 6 gives some suitable filter functions. In Sect. 7 we discuss relationships and differences to the mollified impulse method. Section 8 concludes the paper with numerical experiments on the sine-Gordon equation.

For a recent survey article on existing numerical approaches to oscillatory differential equations we refer to [8].