Fully discrete spectral boundary integral methods for Helmholtz problems on smooth closed surfaces in \( \mathbb{R}^3 \)

I.G. Graham\(^1\), I.H. Sloan\(^2\)

\(^1\) Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, UK; e-mail: I.G.Graham@bath.ac.uk
\(^2\) School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia; e-mail: I.Sloan@unsw.edu.au

Received May 29, 2000 / Revised version received March 26, 2001 / Published online December 18, 2001 – © Springer-Verlag 2001

Summary. In this paper we describe and analyse a class of spectral methods, based on spherical polynomial approximation, for second-kind weakly singular boundary integral equations arising from the Helmholtz equation on smooth closed 3D surfaces diffeomorphic to the sphere. Our methods are fully discrete Galerkin methods, based on the application of special quadrature rules for computing the outer and inner integrals arising in the Galerkin matrix entries. For the outer integrals we use, for example, product-Gauss rules. For the inner integrals, a variant of the classical product integration procedure is employed to remove the singularity arising in the kernel. The key to the analysis is a recent result of Sloan and Womersley on the norm of discrete orthogonal projection operators on the sphere. We prove that our methods are stable for continuous data and superalgebraically convergent for smooth data. Our theory includes as a special case a method closely related to one of those proposed by Wienert (1990) for the fast computation of direct and inverse acoustic scattering in 3D.

Mathematics Subject Classification (1991): 45B05, 65R20

1 Introduction

The aim of this paper is to give a complete stability and convergence analysis of a spectral method for a class of second-kind boundary integral equations.
over the boundary $\partial D$ of a smooth bounded domain $D \in \mathbb{R}^3$. Our equations will be of the form

\begin{equation}
(1.1) \quad v + \mathcal{M}v = f,
\end{equation}

where $\mathcal{M}$ is a linear, weakly singular integral operator on $\partial D$:

\begin{equation}
(1.2) \quad \mathcal{M}v(x) = \int_{\partial D} m(x, y)v(y)ds(y), \quad x \in \partial D.
\end{equation}

The kernel function $m(x, y)$ is allowed to be any linear combination of kernels arising from the classical boundary integral approach to the Helmholtz equation. More precisely, we allow kernels of the form

\begin{equation}
(1.3) \quad m(x, y) = \frac{1}{|x - y|}m_1(x, y) + m_2(x, y)
\end{equation}

(where $|x|$ denotes the Euclidean length of $x$), with $m_i$, for $i = 1, 2$, of the form

\begin{equation}
(1.4) \quad m_i(x, y) = m_{i,1}(x, y) + m_{i,2}(x, y)\frac{(x - y)^Tn(y)}{|x - y|^2} + m_{i,3}(x, y)\frac{(x - y)^Tn(x)}{|x - y|^2},
\end{equation}

with each $m_{i,j}$ infinitely continuously differentiable on $\mathbb{R}^3 \times \mathbb{R}^3$, $i = 1, 2$, $j = 1, 2, 3$, and $n(y)$ denoting the unit outward normal to $\partial D$ at the point $y \in \partial D$.

For example, the single-layer potential for the Helmholtz equation with wave number $k$ has kernel $(1/4\pi) \exp(ik|x - y|)|x - y|^{-1}$. This can be written in the form (1.3), with $m_1(x, y) := (1/4\pi) \cos(k|x - y|)$ and $m_2(x, y) := (1/4\pi) \sin(k|x - y|)|x - y|^{-1}$, which are smooth functions on $\mathbb{R}^3 \times \mathbb{R}^3$.

More generally the representation (1.3), (1.4) includes both the single and double layer potentials for the Helmholtz equation on $\partial D$, together with the normal derivative of the single layer and any linear combination of these operators, all of which appear in formulations of direct [4] and inverse [5, 10, 13] acoustic scattering.

Throughout we will assume that the domain $D$ is connected and has a connected complement $\mathbb{R}^3 \setminus \bar{D}$. We also assume (see (1.6) below) that the surface $\partial D$ is globally parameterised by a smooth (i.e. infinitely continuously differentiable) map $q : \partial B \to \partial D$, where $\partial B = \{\hat{x} \in \mathbb{R}^3 : |\hat{x}| = 1\}$ is the unit sphere. Although this assumption is not satisfied by all conceivable boundaries, it is commonly made in scattering problems (see, e.g. [5, Sect. 3.6]). The chief advantage of the assumption that $\partial D$ has a global