A multilevel discontinuous Galerkin method

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Received June 5, 2001 / Revised version received December 12, 2001 / Published online April 4, 2003 – ©Springer-Verlag 2003

Summary. A variable V-cycle preconditioner for an interior penalty finite element discretization for elliptic problems is presented. An analysis under a mild regularity assumption shows that the preconditioner is uniform. The interior penalty method is then combined with a discontinuous Galerkin scheme to arrive at a discretization scheme for an advection-diffusion problem, for which an error estimate is proved. A multigrid algorithm for this method is presented, and numerical experiments indicating its robustness with respect to diffusion coefficient are reported.

Mathematics Subject Classification (1991): 65F10, 65N55, 65N30

1 Introduction

In this paper we show that a multigrid technique can be used for efficient solution of linear systems arising from the so-called interior penalty finite element method for second order elliptic boundary value problems. We also present a fast method for advection-diffusion equations by combining the interior penalty method with the discontinuous Galerkin method for transport equations, and applying a multigrid technique on the resulting discrete system.

Discontinuous Galerkin (DG) methods have traditionally been used in numerical solution of hyperbolic conservation laws [12,22,24]. Their ability to capture strong gradients in solution without spurious oscillations is well known. Recently DG methods have also been shown to be of use in

This research was supported in part by Institute for Mathematics and its Applications, Supercomputing Institute of University of Minnesota, and Deutsche Forschungsgemeinschaft.
solving elliptic problems [11]. It is now common to classify various earlier methods for solving elliptic problems that went by the name of “interior penalty methods” [2,3,28,13,31] under DG methods (see [1] for a unified treatment). In general, interior penalty finite element methods use discontinuous finite element functions, but penalize discontinuities of the function or its derivatives across inter-element boundaries.

Part of this paper deals with the interior penalty method considered in [2, 31]. When applied to elliptic problems, the method gives rise to linear systems with condition number that grows like $O(h^{-2})$ on quasiuniform grids with mesh size $h$. We prove (in Sect. 3) under weak regularity assumptions that if a variable V-cycle multigrid operator is used to precondition the linear system, then the resulting condition number is $O(1)$, i.e., bounded independently of $h$. The proof is an application of the abstract multigrid theory of [9] for non-inherited bilinear forms. The conjugate gradient method using this preconditioner converges in $O(N)$ operations, where $N$ is the number of unknowns, thus yielding an asymptotically optimal solution technique. In Sect. 4, we confirm and illustrate the theoretical result through numerical experiments.

Preconditioners for discretizations with discontinuous spaces have been studied before [9,29]. In [7], a multigrid analysis for a cell-centered finite difference scheme on uniform grids on square domains is available. Indeed, the interior penalty method can be interpreted as a cell-centered finite difference scheme when piecewise constant functions are used as the discretization space. Our approach here is to use interior penalty estimates directly for multigrid analysis, and we consider spaces of linear or higher order polynomials on more general grids. We note that efficient solution strategies for DG methods can also be constructed using domain decomposition techniques [15,25].

DG schemes show their full potential in advection problems rather than elliptic problems. In Sect. 5, we introduce a DG scheme for an advection-diffusion equation with an arbitrarily small diffusion term. This scheme reduces to the standard DG method for advection problems when the diffusion term is zero. On the other hand, when the advection term is zero, our scheme is the interior penalty method. Before discussing efficient solution strategies for this scheme, we state an error estimate that ensures that it yields good approximate solutions. A proof of this estimate is given in Appendix 6, and may be independently interesting as it provides an error estimate in a slightly stronger norm than some standard estimates (cf. [19]).

For this scheme, we also present a computational technique for fast solution that performs uniformly in both convection dominated and diffusion dominated regimes. This is inspired by the fact that a downwind Gauß–Seidel iteration is an exact solver in the case of zero diffusion. Although this