Non-Abelian Geometry

Keshav Dasgupta, Zheng Yin

School of Natural Sciences, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA.
E-mail: keshav@sns.ias.edu; yin@sns.ias.edu

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Abstract: Spatial noncommutativity is similar and can even be related to the non-Abelian nature of multiple D-branes. But they have so far seemed independent of each other. Reflecting this decoupling, the algebra of matrix valued fields on noncommutative space is thought to be the simple tensor product of constant matrix algebra and the Moyal-Weyl deformation. We propose scenarios in which the two become intertwined and inseparable. Therefore the usual separation of ordinary or noncommutative space from the internal discrete space responsible for non-Abelian symmetry is really the exceptional case of an unified structure. We call it non-Abelian geometry. This general structure emerges when multiple D-branes are configured suitably in a flat but varying B field background, or in the presence of non-Abelian gauge field background. It can also occur in connection with Taub-NUT geometry. We compute the deformed product of matrix valued functions using the lattice string quantum mechanical model developed earlier. The result is a new type of associative algebra defining non-Abelian geometry. A possible supergravity dual is also discussed.

1. Introduction

This paper is devoted to the search and study of certain unusual and hitherto unknown facets of noncommutative space from string and field theories and quantum mechanics. Introducing noncommutativity as a way of perturbing a known field theory has received much interest recently (see [1–3] and the references therein), and hence we shall refrain from repeating the usual motivations and excuses for doing it. Another reason lies in string theory itself. The antisymmetric tensor field $B$ in the Neveu-Schwarz — Neveu-Schwarz sector of string theory, while simpler than it cousins in the Ramond — Ramond sector, is still shrouded in mystery and surprisingly resistant to an unified understanding. One of its many features is its relation to spatial noncommutativity. Let us recall it briefly,
Open strings interact by joining and splitting. This lends naturally to the picture of a geometrical product of open string wave functionals that is clearly noncommutative. One may formulate a field theory of open strings based on this noncommutative product the same way as conventional field theory is formulated on products of wave function fields [4]. But the string wave functional is unwieldy and its product is enormously complex. Noncommutativity certainly does not help. To learn more we have to do with less. One way is to truncate the theory to a low energy effective theory of the small set of massless fields. Another is to approximate the string by a minimal “lattice” of two points. This is especially well suited to mimicking the geometric product of open strings. It emerges from both approximations that, at least using some choice of variables, the natural product of wave function fields is the following noncommutative deformation of the usual one:

\[
(\Psi \ast \Phi)(x) = \exp \left( \frac{i}{2} \frac{\partial}{\partial x'^{\mu}} \Omega^{\mu\nu} \frac{\partial}{\partial x''^{\nu}} \right) \Psi(x')\Phi(x'') \bigg|_{x'=x''=x}.
\] (1.1)

The parameter of noncommutativity \( \Omega \) is expressed in terms of the spacetime metric\(^1 \) \( G \) and \( B \) by

\[
\Omega = -(2\pi\alpha')^2 G^{-1} B G^{-1} \left(1 - (2\pi\alpha')^2 B G^{-1} B G^{-1}\right)^{-1}. \tag{1.2}
\]

It should be noted that noncommutativity is not a consequence of \( B \) being nonvanishing or large. It is intrinsic to the geometry of smooth string junctions that a canonical product exists for the functions on the space of open paths in the target space manifold with the appropriate boundary condition, and that product is noncommutative. The approximations mentioned above induces noncommutativity in the algebra of functions on the submanifold to which the end points are restricted, namely the D-brane. The algebra actually becomes commutative in the limit of very large \( B \)!

It is a glaring deficiency of the present understanding from string theory that one knew only how to deal with a constant and flat \( B \) field. Introducing curvature for \( B \) takes string theory away from the usual sigma model to a rather different realm, so understanding it fully seems to call for a drastic conceptual advance. On the other hand, a varying but flat \( B \) field should be accessible by the available technology but is hampered by technical difficulties. For example, a formal construction of a noncommutative product using an arbitrary Poisson structure in place of the constant \( \Omega \) has been given by Kontsevich [6]. The construction made essential use of a degenerate limit of the sigma model [7]. But the result employs some very abstruse mathematics and its convergence properties are essentially unknown. Behind the complication must lie some interesting and novel structures that need to be deciphered.

We propose, as a first step toward understanding such a situation from string theory, probing it with multiple parallel D-branes configured in a way such that each D-brane only senses a constant but respectively different \( B \) field. On each D-brane the usual noncommutative algebra incorporates the effect of the locally constant \( B \) field without knowing that \( B \) is actually varying. The latter is revealed in the communication among different D-branes via the open strings that start and end on different D-branes. We study the wave functions associated with such “cross” strings and find that their product is deformed in a new and intriguing way that retains associativity. Along the same line of

\[^1\] Note that \( G \) in this paper and in [5] is the same as the “closed string metric” \( g \) in [3], and the noncommutativity parameter \( \Omega \) here is the same as \( \Theta \) there.