Glauber Dynamics of the Random Energy Model

I. Metastable Motion on the Extreme States

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Abstract: We investigate the long-time behavior of the Glauber dynamics for the random energy model below the critical temperature. We give very precise estimates on the motion of the process to and between the states of extremal energies. We show that when disregarding time, the consecutive steps of the process on these states are governed by a Markov chain that jumps uniformly on all possible states. The mean times of these jumps are also computed very precisely and are seen to be asymptotically independent of the terminal point. A first indicator of aging is the observation that the mean time of arrival in the set of states that have waiting times of order $T$ is itself of order $T$. The estimates proven in this paper will furnish crucial input for a follow-up paper where aging is analysed in full detail.

1. Introduction and Background

1.1. Introduction. The concept of “aging” has become one of the main paradigms in the theory of the dynamics of disordered systems\textsuperscript{1}. Roughly speaking, this term refers to a particular way in which dynamic properties of a system change with time when relaxing towards equilibrium: the time scale at which the process evolves slows down in proportion to the elapsed time, the system “ages”. It is in fact believed that most disordered systems, or at least those qualified as “glassy systems” do exhibit this phenomenon. While this is so, almost no results concerning aging in “real” spin systems do exist. In fact most existing results, even on the heuristic level, concern two types of dynamics: 1) Langevin dynamics in spherical models such as the spherical SK model [BDG, CD], or the spherical $p$-spin SK model [BCKM]. 2) Trap models [B, BD, BCKM] that are

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\textsuperscript{1} The cond-mat archives in Trieste contain 263 papers containing this term in their abstracts, and 124 containing it even in the title. See also [Be] for a recent mathematical review.
inspired by the structure of equilibrium states found in (mostly non-rigorous) analysis of mean field spin-glasses. These dynamics are, however, introduced ad hoc without any attempt to justify and derive them from an underlying Glauber dynamics on the microscopic degrees of freedom.

In the context of the spherical models, a rigorous derivation of the aging phenomenon has been given recently in [BDG]. This model lacks, however, many of the expected features of spin glasses, in particular the existence of a complex energy landscape with many “metastable states”. The simplest model showing these features is the random energy model (REM) [D1, D2]. This model is indeed traded as one of the standard examples where aging occurs in the physics literature; the arguments in the physics literature are however, all based on the ad hoc introduction of an effective model (the REM-like trap model [B, BD, BM]) inspired by known properties of the equilibrium distribution and some heuristic arguments. The behaviour of the trap models can then be analysed in detail.

In this and the companion paper [BBG] we prove the first rigorous results on the Glauber dynamics of the REM that will justify in a suitable sense the predictions based on the trap model heuristic. We feel that this is an important first step in showing that the abundant literature on this model is of relevance for realistic disordered systems. The key point of our analysis, and in fact a central problem of the entire subject, will be to control the behaviour of a Markov chain on a very high-dimensional set on a relatively small, but still asymptotically infinite subset of its “most recurrent” or “most stable” states on appropriate time scales, and to describe the ensuing effective dynamics. While we will have to use many of the particular features of the model we consider here, we feel that the general methodology developed in this paper will be of use in many other contexts of the dynamics of complex systems.

The REM. We recall that the REM [D1, D2] is defined as follows. A spin configuration $\sigma$ is a vertex of the hypercube $S_N \equiv \{-1, 1\}^N$. On an abstract probability space $(\Omega, \mathcal{F}, P)$ we define the family of i.i.d. standard normal random variables $\{X_\sigma\}_{\sigma \in S_N}$. We set $E_\sigma = [X_\sigma]_{+} = (X_\sigma \vee 0)$. We define a random (Gibbs) probability measure on $S_N$, $\mu_{\beta,N}$, by setting

$$
\mu_{\beta,N}(\sigma) \equiv \frac{e^{\beta \sqrt{N} E_\sigma}}{Z_{\beta,N}},
$$

where $Z_{\beta,N}$ is the normalizing partition function.\footnote{The standard model has $X_\sigma$ instead of $E_\sigma$. This modification has no effect on the equilibrium properties of the model, and will be helpful for setting up the dynamics.} It is well-known [D1, D2] that this model exhibits a phase transition at $\beta_c = \sqrt{2 \ln 2}$. For $\beta \leq \beta_c$, the Gibbs measures is supported, asymptotically as $N \uparrow \infty$, on the set of states $\sigma$ for which $E_\sigma \sim \sqrt{N} \beta$, and no single configuration has positive mass. For $\beta > \beta_c$, on the other hand, the Gibbs measure gives positive mass to the extreme elements of the order statistics of the family $E_\sigma$; i.e. if we order the spin configurations according to the magnitude of their energies s.t.

$$
E_{\sigma^{(1)}} \geq E_{\sigma^{(2)}} \geq E_{\sigma^{(3)}} \geq \cdots \geq E_{\sigma^{(2N)}},
$$

then for any finite $k$, the respective mass $\mu_{\beta,N}(\sigma^{(k)})$ will converge, as $N$ tends to infinity, to some positive random variable $\nu_k$; in fact, the entire family of masses $\mu_{\beta,N}(\sigma^{(k)})$, $k \in \mathbb{N}$,