Annealed Feynman-Kac Models

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Abstract: We analyze the concentration properties of an annealed Feynman-Kac model in distribution space. We characterize the concentration regions in terms of a variational problem involving a competition between the potential function and the mutation kernel. When the temperature parameter is evanescent with time and under appropriate hypotheses, the probability mass tends to concentrate on regions with minimal potential values. We give a precise description of these areas using non-linear semi-group contractions and large deviation techniques. We illustrate this annealed model with two physical interpretations related respectively to Markov motions in absorbing media and interacting measure valued processes.

1. Introduction

This study is concerned with the long time behavior of a Feynman-Kac model associated to a potential function and a cooling schedule. This annealed distribution flow can be interpreted as the evolution of the laws of a Markov particle in an absorbing medium conditioned to non-extinction. In this context the cooling schedule represents in some sense the temperature of the medium. The smaller it is, the more stringent become the obstacles.

These Feynman-Kac models can alternatively be regarded as a dynamical system in distribution space. In this context they can model the evolution of the marginal laws of a non-linear and non-homogeneous Markov process. The non-linearity comes from the fact that the elementary transitions depend on the distribution flow itself. In this connection the non-linear Markov model can be regarded as a Feynman-Kac type simulated annealing algorithm with mutation/selection transitions. As in the traditional simulated annealing model the temperature parameter is used to increase the selection pressure of the algorithm. These non-linear measure valued processes have a natural genetic type particle interpretation. They have some important applications in biology, advanced signal processing and numerical function analysis [4]. In contrast to previous studies on the
convergence of genetic algorithms in global optimization problems (see for instance [2, 3] and the references given there) we underline that our study is not restricted to finite state spaces and above all that our mutation kernel is homogeneous in time (and not related to the cooling schedule), in particular it does not force the underlying (i.e. unperturbed by potentials) system to be motionless for large time. Our final counter-example relative to convergence to the global minima enables to better apprehend this classical assumption and we will see that it can be advantageously replaced by permitting the mutation kernel to have loops on each point of the state space. This latter precaution is very mild from the point of view of implementation of the procedure, whose speed of convergence should benefit a lot from homogeneity in time of the mutations (and furthermore, endowed with this feature, our algorithm seems closer to true biological/genetic mechanisms than the traditional ones).

The above physical interpretations result from two ways to turn a sub-Markov and Boltzmann-Gibbs operator into a Markov kernel. One of the central questions in the study of these annealed Feynman-Kac models is of course the investigation of the long time behavior of these flows. Intuitively speaking when the temperature parameter tends to zero the probability mass of regions with high potential values decreases and the flow tends to concentrates to regions with minimal potential. The main objective of this article is to make clear this statement. First we discuss the convergence to equilibrium of the annealed models. We exhibit two different types of cooling schedules depending on the mixing parameter of the mutation transitions. Then we characterize the asymptotic regions where the flow concentrates in terms of a variational problem in distribution space. We show that the concentration properties of the annealed model are the result of a competition between the selection potential and the mutation transition. When the temperature parameter tends to zero the variation problem is solved by taking the infimum of the mean potentials over a suitably chosen collection of measures. We already mention that for sufficiently regular mutations and for judicious choices of cooling schedules the annealed model converges in probability to the global infimum of the potential function.

To our knowledge the asymptotic concentration properties of the annealed Feynman-Kac flow presented in this study have not been covered by the literature. We propose an original strategy based on non-linear semi-group contraction and large deviation techniques. The question of the stability of non-linear Feynman-Kac semi-groups arises in many research areas. To our knowledge the first studies in this field originate in non-linear filtering literature. We again refer the reader to [4] for a precise discussion and a precise list of referenced papers. To our knowledge most of these works are only concerned with proving that the flow forgets its initial condition. The reason why these studies do not apply are twofold. First the annealed Feynman-Kac flows presented here are related to an increasing cooling schedule and the resulting functions $e^{-\beta(n)V}$ tend to the indicator function of null potential regions. The analysis of this degenerate situation is more involved and clearly differs from traditional filtering studies. On the other hand our objective is not to check that the flow forgets its initializations but we want to identify the regions on which the distributions concentrate. Our way to enter into this question has been influenced by the article [5]. This study shed some new light on the connections between the limiting distribution of the homogeneous model and the Lyapunov exponent of Schrödinger-Feynman-Kac semi-groups. Here we develop the profound interplay between the asymptotic behavior of these exponents and the concentration properties of the limiting measures. We also present a set of sufficient conditions on the mutation transitions under which these limiting regions coincide with the essential infimum of the potential function with respect to the invariant measure of the mutation kernel. We end