Spectral Properties of Hypoelliptic Operators

J.-P. Eckmann\textsuperscript{1,2}, M. Hairer\textsuperscript{1,*}

\textsuperscript{1} Département de Physique Théorique, Université de Genève, Geneve, Switzerland.
E-mail: Jean-Pierre.Eckmann@physics.unige.ch; Martin.Hairer@physics.unige.ch
\textsuperscript{2} Section de Mathématiques, Université de Genève, Geneve, Switzerland

Received: 30 July 2002 / Accepted: 18 October 2002
Published online: 25 February 2003 – © Springer-Verlag 2003

Abstract: We study hypoelliptic operators with polynomially bounded coefficients that are of the form \( K = \sum_{i=1}^{m} X_i^T X_i + X_0 + f \), where the \( X_j \) denote first order differential operators, \( f \) is a function with at most polynomial growth, and \( X_i^T \) denotes the formal adjoint of \( X_i \) in \( L^2 \). For any \( \varepsilon > 0 \) we show that an inequality of the form
\[
\| u \|_{\delta,\delta} \leq C(\| u \|_{0,\varepsilon} + \| (K+iy)u \|_{0,0})
\]
holds for suitable \( \delta \) and \( C \) which are independent of \( y \in \mathbb{R} \), in weighted Sobolev spaces (the first index is the derivative, and the second the growth). We apply this result to the Fokker-Planck operator for an anharmonic chain of oscillators coupled to two heat baths. Using a method of Hérau and Nier [HN02], we conclude that its spectrum lies in a cusp
\[
\{ x + iy | x \geq \vert y \vert^\tau - c, \tau \in (0,1], c \in \mathbb{R} \}
\]

1. Introduction

In an interesting paper, [HN02], Hérau and Nier studied the Fokker-Planck equation associated to a Hamiltonian system \( H \) in contact with a heat reservoir at inverse temperature \( \beta \). For this problem, it is well-known that the Gibbs measure
\[
\mu_\beta(dp\,dq) = \exp(-\beta H(p,q))\,dp\,dq
\]
is the only invariant measure for the system. In their study of convergence under the flow of any measure to the invariant measure, they were led to study spectral properties of the Fokker-Planck operator \( \mathcal{L} \) when considered as an operator on \( L^2(\mu_\beta) \). In particular, they showed that \( \mathcal{L} \) has a compact resolvent and that its spectrum is located in a cusp-shaped region, as depicted in Fig. 1.1 below, improving (for a special case) earlier results obtained by Rey-Bellet and Thomas [RBT02b], who showed that \( e^{-\mathcal{L}t} \) is compact and that \( \mathcal{L} \) has spectrum only in \( \text{Re} \lambda > c > 0 \) aside from a simple eigenvalue at 0.

\textsuperscript{*} Present address: Mathematics Research Centre of the University of Warwick
Extending the methods of [HN02], we show in this paper that the cusp-shape of the spectrum of $\mathcal{L}$ occurs for many Hörmander-type operators of the form

$$K = \sum_{i=1}^{m} X_i^T X_i + X_0 + f,$$  \hspace{1cm} (1.1)  

(the symbol $^T$ denotes the formal $L^2$ adjoint) when the family of vector fields $\{X_j\}_{j=0}^{m}$ is sufficiently non-degenerate (see Definition 2.1 and assumption $b_1$ below) and some growth condition on $f$ holds.

The main motivation for our paper comes from the study of the model of heat conduction proposed in [EPR99a] and further studied in [EPR99b, EH00, RBT00, RBT02b, RBT02a]. These papers deal with Hamiltonian anharmonic chains of point-like particles with nearest-neighbor interactions whose ends are coupled to heat reservoirs modeled by linear classical field theories. Our results improve the detailed knowledge about the spectrum of the generator $\mathcal{L}$ of the associated Markov process, see Sect. 5. As a by-product, our paper also gives a more elegant analytic proof of the results obtained in [EH00]. A short probabilistic proof has already been obtained in [RBT02b].

The main technical result needed to establish the cusp-form of the spectrum is the Sobolev estimate Theorem 4.1 which seems to be new.

2. Setup and Notations

We will derive lower bounds for hypoelliptic operators with polynomially bounded coefficients that are of the form (1.1). We start by defining the class of functions and vector fields we consider.

2.1. Notations. For $N \in \mathbb{R}$, we define the set $\text{Pol}_0^N$ of polynomially growing functions by

$$\text{Pol}_0^N = \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^{-N} |\partial^\alpha f(x)| \leq C_\alpha \right\}.  \hspace{1cm} (2.1)$$