Conformal Subnets and Intermediate Subfactors

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Dedicated to Rudolf Haag on the occasion of his eightieth birthday

Abstract: Given an irreducible local conformal net \(A\) of von Neumann algebras on \(S^1\) and a finite-index conformal subnet \(B \subset A\), we show that \(A\) is completely rational iff \(B\) is completely rational. In particular this extends a result of F. Xu for the orbifold construction. By applying previous results of Xu, many coset models turn out to be completely rational and the structure results in [27] hold. Our proofs are based on an analysis of the net inclusion \(B \subset A\); among other things we show that, for a fixed interval \(I\), every von Neumann algebra \(R\) intermediate between \(B(I)\) and \(A(I)\) comes from an intermediate conformal net \(\mathcal{L}\) between \(B\) and \(A\) with \(\mathcal{L}(I) = R\). We make use of a theorem of Watatani (type II case) and Teruya and Watatani (type III case) on the finiteness of the set \(\mathcal{I}(N, M)\) of intermediate subfactors in an irreducible inclusion of factors \(N \subset M\) with finite Jones index \([M : N]\). We provide a unified proof of this result that gives in particular an explicit bound for the cardinality of \(\mathcal{I}(N, M)\) which depends only on \([M : N]\).

1. Introduction

Operator algebraic methods have been used to good effect in Conformal Quantum Field Theory, in particular in understanding general model independent structure (e.g. [6, 16, 22, 23, 27, 38]), in the analysis of concrete models (e.g. [5, 43, 44, 46]) and for applications in different contexts (e.g. [33]). In most cases it seems to be impossible to proceed by different methods.

Because of their relevance in different areas, among others Topological QFT and 3-manifold invariants, conformal models with a rational and modular representation theory have been the subject of much attention, also in the physical literature (cf. [14]).

In [27] intrinsic, model independent conditions selecting a class of (local, irreducible) conformal nets \(A\) of von Neumann algebras on \(S^1\) with the right rationality/modularity properties were given. \(A\) is completely rational if

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1. $A$ is split, 
2. $A$ is strongly additive, 
3. the 2-interval inclusion of factors $A(E) \subset A(E')$ has finite Jones index $\mu_A$.

Here both $E \subset S^1$ and $E' \equiv S^1 \setminus E$ are the union of two proper intervals. The split and strongly additivity properties are well-studied basic properties, see Sect. 3.5 for their definitions, and we do not dwell on them here, cf. [13, 7, 11] and [8, 23]. If $A$ is completely rational, then $A(E) \subset A(E')$ is obtained by a quantum double construction in [34], in particular

$$\mu_A = \sum_i d(\rho_i)^2,$$

where the sum is taken over all the irreducible sectors of $A$. Every representation of $A$ (on a separable Hilbert space) is Möbius covariant and decomposes into the direct sum of irreducible representations with finite statistical dimension. There are only finitely many inequivalent irreducible representations, i.e. $A$ is rational, and the associated braiding is non-degenerate, i.e. the representation tensor category is modular.

At this point the problem of verifying the complete rationality of known models arises. Certain examples were discussed in [27]. As an illustration from [27], consider the case of a non-trivial finite group $G$ acting on a completely rational $A$; if the fixed-point orbifold subnet $A^G$ is also completely rational, then $\mu_{A^G} \geq |G|^2$, while $\sum_{\pi \in \hat{G}} d(\rho_\pi)^2 = |G|$, where the $\rho_\pi$’s are the untwisted DHR sectors of $A^G$ [12], and this shows that twisted sectors must appear. As $A$ is the initial data, one would infer the complete rationality of $A^G$ from that of $A$. By [27] $A^G$ inherits from $A$ the split property and the finiteness of the $\mu$-index. F. Xu [46] has then shown that $A^G$ also inherits the strong additivity property and this has inspired our paper.

We shall now show that if $B$ is any conformal subnet of $A$ with finite index, then $B$ is completely rational iff $A$ is completely rational.

As a consequence, if $B$ is a cofinite subnet of $A$, namely $[A : B \vee B^c] < \infty$, where $B \vee B^c$ is the subnet generated by $B$ and its relative commutant $B^c$ in $A$, then $A$ is completely rational iff both $B$ and $B^c$ are completely rational.

The subnet $B^c$ is called the coset subnet associated with $B \subset A$, as it generalizes a coset construction that plays an important rôle in the theory of Kac-Moody Lie algebras, allowing one to construct the minimal series representations of the Virasoro algebra [19].

Coset models have been intensively studied by Xu in [43, 44] by operator algebraic methods. In one approach he makes use of [27] too. Thanks to his work, coset models associated with many loop group inclusions are cofinite, rational and modular, see the list in Sect. 3.5.1. Property 3 holds, but the validity of strong additivity was left open.

By our work in all these examples $B^c$ is strongly additive, thus $B^c$ turns out to be completely rational and this completes the above discussion and explains the rationality/modularity structure better.

We now comment on our proof that the complete rationality property (and also the “split & strongly additivity” property) for finite-index inclusions of conformal nets $B \subset A$ are hereditary. That “Property 1 & 3” passes from $A$ to $B$ and vice versa is shown in [27]. The remaining more difficult point we have to prove is that $B$ is strongly additive if $A$ is split and strongly additive, see Sect. 3.5.

To this end, we have analyzed a finite-index inclusion of conformal nets $B \subset A$ by considering the relative superselection structure. In particular we show that, for a fixed