On the Renormalization Group in Curved Spacetime

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Abstract: We define the renormalization group flow for a renormalizable interacting quantum field in curved spacetime via its behavior under scaling of the spacetime metric, \( g \rightarrow \lambda^2 g \). We consider explicitly the case of a scalar field, \( \varphi \), with a self-interaction of the form \( \kappa \varphi^4 \), although our results should generalize straightforwardly to other renormalizable theories. We construct the interacting field – as well as its Wick powers and their time-ordered-products – as formal power series in the algebra generated by the Wick powers and time-ordered-products of the free field, and we determine the changes in the interacting field observables resulting from changes in the renormalization prescription.

Our main result is the proof that, for any fixed renormalization prescription, the interacting field algebra for the spacetime \((M, \lambda^2 g)\) with coupling parameters \( p \) is isomorphic to the interacting field algebra for the spacetime \((M, g)\) but with different values, \( p(\lambda) \), of the coupling parameters. The map \( p \rightarrow p(\lambda) \) yields the renormalization group flow.

The notion of essential and inessential coupling parameters is defined, and we define the notion of a fixed point as a point, \( p \), in the parameter space for which there is no change in essential parameters under renormalization group flow.

1. Introduction

Theories of a classical field in Minkowski spacetime that are derived from an action principle will automatically possess an invariance under a scaling of the global inertial coordinates of spacetime (or, equivalently, under scaling of the field momenta) provided that a corresponding scaling of the field amplitude and coupling constants are also performed in such a way that the action remains unchanged. If the quantum theory of this field is renormalizable, it turns out that in perturbation theory there also is a similar invariance of quantities of interest – such as the Green’s functions of the fields – under scaling of the field momenta, but the required scaling of the field amplitudes and coupling constants differs, in general, from the simple scaling laws for the classical theory. This change of the “field strength normalization” and coupling constants under scaling
is called the “renormalization group flow” of the theory. Important qualitative as well as quantitative information about quantum field theories can be gained from an analysis of their renormalization group flow.

For quantum field theories in Minkowski spacetime, there exist well known procedures for calculating the renormalization group flow in perturbation theory. In many cases, the picture obtained from low orders is believed to be at least in qualitative agreement with the behavior that would hold in the full, nonperturbatively constructed quantum field theory. Consequently, perturbative calculations of the renormalization group flow have played an important role in arguments concerning fundamental properties of quantum field theories. In particular, they form the basis of the claim that certain non-abelian gauge theories are “asymptotically free”, i.e., that the gauge coupling flows towards zero at small distances (large momenta).

It is therefore of interest to know whether a similar scaling analysis can also be performed for perturbative interacting quantum field theory on an arbitrary globally hyperbolic curved (Lorentzian) spacetime. As we shall briefly review in Sect. 2 below, the construction of perturbative interacting quantum field theory in curved spacetime has recently been achieved in [15, 16], based upon some earlier key results established in [3, 4] and other references. However, for at least the following two reasons, it does not seem possible to give a straightforward generalization to curved spacetime of the usual scaling analyses given for Minkowski spacetime. First, as already indicated above, the renormalization group flow in Minkowski spacetime is usually formulated in terms of behavior under the scaling of global inertial coordinates or, equivalently, scaling of the field momenta. However, in curved spacetime a formulation in terms of scaling of coordinates (or momenta) would introduce a very awkward and undesired coordinate dependence into the constructions. Also, since the scaling of coordinates no longer corresponds to a conformal isometry of the spacetime metric, one would not expect a simple behavior to occur under scalings of any coordinates. Second, the quantities whose scaling behavior is usually considered in studying the renormalization group flow in Minkowski space are the Green’s functions of the interacting field or other quantities from which these can be derived, such as the “effective action”. However, the Green’s functions depend on a choice of state. For quantum field theories in Minkowski spacetime, this state would naturally be chosen to be the (unique) Poincaré invariant vacuum state. However, even for a free quantum field in a general curved spacetime, there is no “preferred vacuum state” nor any other state that can be singled out for special consideration. Thus, even if a renormalization group flow could be defined in terms of Green’s functions, there is no reason to expect it to be independent of the choice of state used to define the Green’s functions.

A solution to the second difficulty is achieved by formulating the theory via the algebraic approach. In this approach, one views the observables as forming an abstract algebra, and one views the quantum states as suitable linear functionals on this algebra. This algebra is referred to as “abstract”, because no representation of this algebra on a particular Hilbert space has been chosen from the outset, so that the (potentially problematic) issue of choosing states is completely disentangled from the issue of constructing the observables of the theory. As we shall see, the renormalization group flow can then be defined at the level of the algebra of observables\(^1\).

The first difficulty above is solved by defining the renormalization group flow in terms of the behavior of the algebra of the interacting field under a scaling of the spacetime

\(^{1}\) An approach very different from ours that also treats the renormalization group flow at the level of the algebra of observables is given in [6].