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Twisted Orbifold K-Theory

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Abstract: We use equivariant methods to define and study the *orbifold K-theory* of an orbifold X. Adapting techniques from equivariant K-theory, we construct a Chern character and exhibit a multiplicative decomposition for $K^*_{orb}(X) \otimes \mathbb{Q}$, in particular showing that it is additively isomorphic to the orbifold cohomology of X. A number of examples are provided. We then use the theory of projective representations to define the notion of twisted orbifold K–theory in the presence of discrete torsion. An explicit expression for this is obtained in the case of a global quotient.

Contents

1. Motivation

During the last twenty years, one major trend has been the constant flow of physical ideas into mathematics. Although many of them have had a significant impact, there are still many others which deserve more attention from mathematicians. Orbifold string theory is such an example: it has been around in physics since 1985; compared to its

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popularity in physics, its influence in mathematics is minimal. This project represents an effort to change this picture.

First, let us give a brief introduction to orbifold string theory aiming to explain its mathematical content. In the end, we hope that it will become clear why we have to study orbifold K-theory. In 1985, Dixon-Harvey-Vafa-Witten [16] discovered that on a singular manifold such as an orbifold one can also build a smooth consistent string theory. One of the remarkable insights of orbifold string theory is that its consistency demands the introduction of so-called twisted sectors. In other words, ordinary cohomology is the **wrong** theory for orbifolds; the correct one seems to be *orbifold cohomology*, which has been constructed by Chen-Ruan [13]. Orbifold cohomology (not ordinary cohomology) should fulfill the role of cohomology for smooth manifolds. Moreover, on smooth manifolds, orbifold cohomology is the same as ordinary cohomology.

One of the original motivations of orbifold string theory is to capture the information of a *crepant resolution* (see [21] p. 126 for definitions). Physically, the string theories corresponding to an orbifold and its crepant resolution are in a family of string theories. To compare them, one has to deform the string theory from one point to another. It is known that the orbifold points are often singularities. Its monodromy could interchange $H^{1,1}$ and $H^{2,2}$. Therefore, there is no hope to preserve the grading for an invariant such as orbifold cohomology. However, it still preserves the *parity*. This naturally motivates the concept of *orbifold K-theory*[1] which we shall define in Sect. 4. Namely, orbifold K-theory (not orbifold cohomology) is better suited for the relation between an orbifold and its crepant resolution. Once we have defined an appropriate orbifold K-theory K_{orb}, a natural conjecture arising from orbifold string theory is the

K-Orbifold String Theory Conjecture. *If X is a complex orbifold and $Y \to X$ is a crepant resolution, then there is a natural additive isomorphism $K(Y)\otimes\mathbb{C} \cong K_{orb}(X)\otimes\mathbb{C}$, between the orbifold K-theory of X and the ordinary K-theory of its crepant resolution Y.*

Note for example that if X is a complex 3–orbifold with orbifold groups in $\mathrm{SL}_3(\mathbb{C})$, then it admits a crepant resolution–this condition is automatically satisfied by Calabi–Yau orbifolds.

Another physical motivation comes from the recent discovery in physics that the D-brane charge of an ordinary string theory is described by the K-theory of its underlying smooth manifold. For orbifold string theory, it is obvious that the orbifold D-brane charge should be described by our orbifold K-theory.

An obvious definition of orbifold K-theory is the K-theory associated to orbifold vector bundles. However, to be a correct definition for our purposes, we have to obtain at least an additive isomorphism to orbifold cohomology. It is well-known that any reduced orbifold can be expressed as the quotient of a smooth manifold by an almost free action of a compact Lie group (see Sect. 2). Therefore, we will use methods from equivariant topology to provide an effective K-theoretic approach to orbifolds. In particular, using an appropriate equivariant Chern character we obtain a decomposition theorem for *orbifold K-theory* as a ring, and we show that it is additively isomorphic to orbifold cohomology. A nice by-product of our orbifold K-theory is a natural notion of orbifold Euler number for a general orbifold. In the context of orbifold cohomology, it only makes sense for SL-orbifolds, where the local group is a finite subgroup of $SL(n, C)$.

[1] Throughout this paper we shall be dealing exclusively with (equivariant) complex K–theory, i.e. the generalized cohomology theory arising from complex vector bundles.