Critical \((\Phi^4)_3, \epsilon\)

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Abstract: The Euclidean \((\phi^4)_3, \epsilon\) model in \(\mathbb{R}^3\) corresponds to a perturbation by a \(\phi^4\) interaction of a Gaussian measure on scalar fields with a covariance depending on a real parameter \(\epsilon\) in the range \(0 \leq \epsilon \leq 1\). For \(\epsilon = 1\) one recovers the covariance of a massless scalar field in \(\mathbb{R}^3\). For \(\epsilon = 0\), \(\phi^4\) is a marginal interaction. For \(0 \leq \epsilon < 1\) the covariance continues to be Osterwalder-Schrader and pointwise positive. We consider the infinite volume critical theory with a fixed ultraviolet cutoff at the unit length scale and we prove that for \(\epsilon > 0\), sufficiently small, there exists a non-gaussian fixed point (with one unstable direction) of the Renormalization Group iterations. We construct the stable critical manifold near this fixed point and prove that under Renormalization Group iterations the critical theories converge to the fixed point.

1. Introduction, Model, RG Transformation

1.1. Introduction. Let \(\phi\) be a mean zero Gaussian scalar random field on \(\mathbb{R}^3\) with covariance

\[
E(\phi(x)\phi(y)) = \frac{\text{const}}{|x-y|^{(3-\epsilon)/2}} = \int \frac{d^3p}{(2\pi)^3} e^{i(p\cdot(x-y))} (p^2)^{-(3+\epsilon)/4}. \tag{1.1}
\]

Here \(p^2 = |p|^2\) is the standard Euclidean norm in \(\mathbb{R}^3\) and \(p \cdot x\) is the standard scalar product. \(\epsilon\) is a real parameter which we take in the region \(0 \leq \epsilon \leq 1\). Note that for \(\epsilon = 1\) we have the standard massless free scalar field in \(\mathbb{R}^3\). This covariance \((-\Delta)^{-(3+\epsilon)/4}\) has interesting properties. In particular, scaling limits of suitable perturbations (see below) of these theories should be Euclidean quantum field theories, because \((-\Delta)^{-(3+\epsilon)/4}\) is...
Osterwalder-Schrader positive not only for $\epsilon = 1$ (see [GJ]) but also for $0 \leq \epsilon < 1$. The latter follows from the convergent integral representation

$$(-\Delta)^{-(3+\epsilon)/4}(x - y) = 1/c_\epsilon \int_0^\infty ds s^{-(3+\epsilon)/4}(-\Delta + s)^{-1}(x - y), \quad (1.2)$$

where $c_\epsilon = \int_0^\infty ds s^{-(3+\epsilon)/4}(1 + s)^{-1}$ and $0 \leq \epsilon < 1$. The scaling limit should be constructible and it will not be a generalized free field, because the result in this paper shows that there is a non-Gaussian Infrared fixed point.

From the same integral representation one also finds that the kernel of $(-\Delta)^{-(3+\epsilon)/4}$ is pointwise positive. Furthermore $(-\Delta)^{-(3+\epsilon)/4}$ is the Green’s function (or potential) for a stable Lévy process in $\mathbb{R}^3$ with parameters $(\alpha, \beta)$ in the Lévy-Khintchine notation [KG], with the characteristic exponent $\alpha = (3+\epsilon)/2$, and $\beta = 0$. The sample paths have discontinuities and the process diffuses very fast. We hope, in the future, to exploit this to study self-avoiding Levy processes by representing them as supersymmetric versions of the functional integral of this paper, following the program set out in [BI].

These properties make the study of the above gaussian random field and its non-linear perturbations worthwhile. In particular we will study here the critical properties of a model corresponding to the partition function

$$Z = \int d\mu_C(\phi) e^{-V_0(\phi)}, \quad (1.3)$$

where $d\mu_C$ is the Gaussian measure with covariance $C$ and $C$ is $(-\Delta)^{-(3+\epsilon)/4}$ with a fixed unit scale ultraviolet cutoff described below and

$$V_0(\phi) = V(\phi, C, g_0, \mu_0) = g_0 \int d^3 x : \phi^4 :_C (x) + \mu_0 \int d^3 x : \phi^2 :_C (x) \quad (1.4)$$

and the coupling constant $g$ is held strictly positive. Moreover in order to define the model completely we must also introduce a volume cutoff. However as soon as we have described how to parametrise the theories that comprise the domain of the Renormalization Group, there is a clear definition and candidate for the infinite volume limit and that is the object under consideration in this paper. These cutoffs will be introduced presently when we give a precise definition of the covariance $C$ in (1.4) as well as that of the model.

From (1.1) we can read off the canonical scaling dimension $[\phi]$ of $\phi$: $[\phi] = (3 - \epsilon)/4$. This, together with (1.3), implies that we can assign to $g, \mu$ the dimensions $[g] = \epsilon, \ [\mu] = (3 + \epsilon)/2$. Note that we have not put in a term $\frac{1}{2} \int d^3 x |\nabla \phi(x)|^2$ in (1.4). This is because the dimension $[\epsilon] = -(1 - \epsilon)/2$. Hence for $\epsilon < 1$ this is a candidate for an irrelevant (stable) direction.

The point of this paper is to prove that Wilson’s theory of critical phenomena [KW] is correct for this model: namely, that for $\epsilon > 0$ the critical (infra-red) properties of the model are dominated by a non-Gaussian fixed point of Renormalization Group iterations with $g = g_\epsilon = O(\epsilon)$ provided the unstable parameter $\mu$ is fine tuned to a critical function $\mu_c(g)$ which determines the stable (critical) manifold of the fixed point. In the present work we will prove the existence of the non-Gaussian fixed point for $\epsilon > 0$ held sufficiently small and, on the way, construct the stable manifold. There are open problems, which may be accessible to these methods. The first is to verify that the fixed point describes a scaling limit which satisfies the Osterwalder-Schrader axioms (see