Classification of Harish-Chandra Modules over the Higher Rank Virasoro Algebras*

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Abstract: We classify the Harish-Chandra modules over the higher rank Virasoro and super-Virasoro algebras: It is proved that a Harish-Chandra module, i.e., an irreducible weight module with finite weight multiplicities, over a higher rank Virasoro or super-Virasoro algebra is either a module of the intermediate series, or a finitely-dense module. As an application, it is also proved that an indecomposable weight module with finite weight multiplicities over a generalized Witt algebra is either a uniformly bounded module (i.e., a module with weight multiplicities uniformly bounded) with all nonzero weights having the same multiplicity, or a finitely-dense module, as long as the generalized Witt algebra satisfies one minor condition.

1. Introduction

The notions of the higher rank Virasoro algebras [13] and the higher rank super-Virasoro algebras [17] appear as natural generalizations of the well-known Virasoro algebra [4] and super-Virasoro algebras (the Neveu-Schwarz superalgebra [12], the Ramond superalgebra [14]). As the universal central extension of the complex Lie algebra of the linear differential operators over the circle (the Witt algebra), the Virasoro algebra \( \mathfrak{Vir} \), closely related to Kac-Moody algebras [5–7], is of much interest to both mathematicians and physicists, partly due to its relevance to string theory [15] and 2-dimensional conformal field theory [3]. The Virasoro algebra \( \mathfrak{Vir} \) can be defined as a Lie algebra with basis \( \{ L_i, c | i \in \mathbb{Z} \} \) such that \( [L_i, L_j] = (j - i)L_{i+j} + \frac{j^3 + j}{12} \delta_{i,-j} c, [L_i, c] = 0 \) for all \( i, j \in \mathbb{Z} \). Let \( n \) be a positive integer. Let \( M \) be an \( n \)-dimensional \( \mathbb{Z} \)-submodule of \( \mathbb{C} \), and let \( s \in \mathbb{C} \), be such that \( 2s \in M \). A rank \( n \) Virasoro algebra (or a higher rank Virasoro algebra if \( n \geq 2 \)) [13] is a complex Lie algebra \( \mathfrak{Vir}[M] \) with basis \( \{ L_\mu, c | \mu \in M \} \), such that

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\[ [L_{\mu}, L_v] = (v - \mu) L_{\mu + v} - \frac{\mu^3 - \mu}{12} \delta_{\mu,-v} c, \quad [L_{\mu}, c] = 0, \quad \forall \mu, v \in M. \quad (1.1) \]

A rank \( n \) super-Virasoro algebra (or a higher rank super-Virasoro algebra if \( n \geq 2 \)) \cite{17} is the Lie superalgebra \( \text{SVir}[M, s] = \text{SVir}_0 \oplus \text{SVir}_1 \), where \( \text{SVir}_0 = \text{Vir}[M] \) has a basis \( \{ L_{\mu}, c | \mu \in M \} \) and \( \text{SVir}_1 \) has a basis \( \{ G_{\eta} | \eta \in s + M \} \), with the commutation relations \((1.1)\) and
\[ [L_{\mu}, G_\eta] = (\eta - \frac{\mu}{2}) G_{\mu + \eta}, \quad [G_\eta, G_\lambda] = 2L_{\eta + \lambda} - \delta_{\eta,-\lambda} \left( \frac{1}{2}(\eta^2 - \frac{1}{2})c \right), \quad [G_\eta, c] = 0, \quad \forall \mu \in M, \quad \eta, \lambda \in s + M. \]

The irreducible representations of the Virasoro and super-Virasoro algebras, which play very important roles in the theory of vertex operator (super)algebras and mathematical physics, are well developed (see for example \cite{2, 6–9, 16, 18}). However for the higher rank case, not much has been known except modules of the intermediate series \cite{17, 19, 20}, or Verma-like modules over the centerless higher rank Virasoro algebras \cite{10}.

A weight module is a module \( V \) with weight space decomposition:
\[ V = \bigoplus_{\lambda \in \mathbb{C}^2} V_{\lambda}, \quad V_{\lambda} = \{ v \in V | L_0 v = \lambda_0 v, c v = h v \}, \quad \text{where} \quad \lambda = (\lambda_0, h) \in \mathbb{C}^2. \]

A weight module is quasi-finite if all weight spaces \( V_{\lambda} \) are finite dimensional. A module of the intermediate series is an indecomposable quasi-finite weight module with all the dimensions of weight spaces (and in the super case all the dimensions of weight spaces of “even” or “odd” part) are \( \leq 1 \). It is proved \cite{17} that a module of the intermediate series over a higher rank Virasoro algebra \( \text{Vir}[M] \) is \( A_{a,b}, A_{a'}, B_{a'} \) or one of their quotients for suitable \( a, b, a' \in \mathbb{C} \), where \( A_{a,b}, A_{a'}, B_{a'} \) all have a basis \( \{ x_{\mu} | \mu \in M \} \) such that \( c \) acts trivially and
\[ A_{a,b} : L_{\mu} x_{\nu} = (a + v + \mu b) x_{\mu + v}, \quad (1.2a) \]
\[ A_{a'} : L_{\mu} x_{\nu} = (v + \mu) x_{\mu + v}, \quad (1.2b) \]
\[ B_{a'} : L_{\mu} x_{\nu} = v x_{\mu + v}, \quad (1.2c) \]

for \( \mu, v \in M \); and that a module of the intermediate series over the higher rank super-Virasoro algebras \( \text{SVir}[M, s] \) is one of the three series of the modules \( S A_{a,b}, S A_{a'}, S B_{a'} \) or their quotient modules for suitable \( a, b, a' \in \mathbb{C} \), where \( S A_{a,b}, S A_{a'} \) have basis \( \{ x_{\mu} | \mu \in M \} \cup \{ y_{\eta} | \eta \in s + M \} \) and \( S B_{a'} \) has basis \( \{ x_{\eta} | \eta \in s + M \} \cup \{ y_{\mu} | \mu \in M \} \) such that \( c \) acts trivially and
\[ S A_{a,b} : L_{\mu} x_{\nu} = (a + v + \mu b) x_{\mu + v}, \quad L_{\mu} y_{\eta} = (a + \eta + \mu(b - \frac{1}{2})) y_{\mu + \eta}, \]
\[ G_{\lambda} x_{\nu} = y_{\lambda + v}, \quad G_{\lambda} y_{\eta} = (a + \eta + 2\lambda(b - \frac{1}{2})) x_{\lambda + \eta}, \]
\[ S A_{a'} : L_{\mu} x_{\nu} = (v + \mu) x_{\mu + v}, \quad L_{\mu} y_{\eta} = (\eta + \frac{\mu}{2}) y_{\mu + \eta}, \]
\[ G_{\lambda} x_{\nu} = y_{\lambda + v}, \quad G_{\lambda} y_{\eta} = (\eta + 2\lambda)x_{\lambda + \eta}, \]
\[ S B_{a'} : L_{\mu} x_{\nu} = (\eta + \frac{\mu}{2}) x_{\mu + \eta}, \quad L_{\mu} y_{\nu} = v y_{\mu + v}, \quad v \neq -\mu, \quad L_{\mu} y_{-\mu} = -\mu(y + \mu') y_{\nu}, \]
\[ G_{\lambda} x_{\nu} = y_{\lambda + v}, \quad \eta \neq -\lambda, \quad G_{\lambda} x_{-\lambda} = (2\lambda + a') y_{\lambda}, \quad G_{\lambda} y_{\nu} = v y_{\lambda + v}, \]

for \( \mu, v \in M, \lambda, \eta \in s + M \). An uniformly bounded module is a quasi-finite weight module with all weight multiplicities being uniformly bounded. It is proved \cite{19, 20} that a uniformly bounded irreducible