Superconformal Symmetry and HyperKähler Manifolds with Torsion

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Abstract: The geometry arising from Michelson & Strominger’s study of $\mathcal{N} = 4B$ supersymmetric quantum mechanics with superconformal $D(2, 1; \alpha)$-symmetry is a hyperKähler manifold with torsion (HKT) together with a special homothety. It is shown that different parameters $\alpha$ are related via changes in potentials for the HKT target spaces. For $\alpha \neq 0, -1$, we describe how each such HKT manifold $M^4m$ is derived from a space $N^4m - 4$ which is quaternionic Kähler with torsion and carries an Abelian instanton.

1. Introduction

In the study of two-dimensional sigma models a variety of different quaternionic geometries arise on the target spaces. In the presence of a Wess-Zumino term the metric connections have non-zero torsion. For $\mathcal{N} = 4B$ rigid supersymmetry the target space carries an HKT structure: the geometry of a hyperKähler connection with totally skew symmetric torsion [4]. For $\mathcal{N} = 4B$ local symmetry the resulting geometry [6] is known as QKT (quaternionic Kähler with torsion). The mathematical background of HKT geometry was reported in [5], where many examples were constructed. Mathematical discussion of QKT geometry may be found in [7].

Through the work of Maldacena [8] there has been much interest in field theories with superconformal symmetry. Michelson and Strominger [9] showed that for $\mathcal{N} = 4B$ rigid supersymmetry examples of quantum mechanical systems in one dimension with actions of the superconformal groups $D(2, 1; \alpha)$ may be obtained. As discussed in [2], $D(2, 1; \alpha)$ has $\text{su}(2) \oplus \text{su}(2)$ as its algebra of $R$-symmetries and $D(2, 1; -2)$ is the supergroup $\text{Osp}(4|2)$. On the target space, Michelson and Strominger [9] show that the HKT manifold (locally) has a certain vector field $X$ generating one homothety and three isometries, see Eqs. (2.2). In this paper we investigate the geometry of an HKT manifold with such a vector field. In [12], we showed that the length-squared of $X$ gives a
potential \( \mu \) for the HKT metric. By transforming \( \mu \) we show in Sect. 3 that \( D(2, 1; \alpha) \)-symmetries for different values of \( \alpha \) are closely related. In particular, if an HKT manifold has a vector field \( X \) generating a \( D(2, 1; \alpha) \)-symmetry with \( \alpha < 0 \) and \( \alpha \neq -1 \), then the same manifold carries HKT metrics with \( D(2, 1; \alpha') \)-symmetry for each \( \alpha' < 0 \). Similarly, any \( \alpha' > 0 \) may be obtained from any other \( \alpha > 0 \).

In Sect. 4 we show that the vector fields generate an infinitesimal action of the non-zero quaternions \( \mathbb{H}^* \) and that the quotient \( \mathcal{N}^4n = M/\mathbb{H}^* \) carries a QKT metric. It turns out, Sect. 5, that this QKT manifold comes equipped with an instanton connection on its bundle \( \Lambda^* \mathcal{N}^4n T^* \) of volume forms. Locally QKT metrics inducing instanton connections exist on any quaternionic manifold, and from such a geometry in dimension \( 4n \) we construct in Sect. 6 HKT metrics with \( D(2, 1; \alpha) \)-symmetry in dimension \( 4n + 4 \). As an interesting special case, we obtain HKT metrics with \( D(2, 1; 1) \)-symmetry over each quaternionic Kähler manifold of negative scalar curvature.

Both the discussion of the parameter change for \( D(2, 1; \alpha) \)-symmetry and the bundle constructions relating QKT and HKT geometries naturally introduce pseudo-Riemannian structures. We therefore deal with HKT geometry in this generality from the outset.

If one sets the torsion to zero in this paper, then one recovers the constructions of [13], relating quaternionic Kähler manifolds to hyperKähler manifolds with \( D(2, 1; -2) \)-symmetry and hyperKähler potentials. This case is relevant to the discussion of superconformal symmetry in \( N = 2 \) quantum mechanics [3].

2. Potentials and Superconformal Symmetry

Let \((M, g, I, J, K)\) be an HKT manifold of dimension \( 4m \) and signature \((4p, 4q)\). This means that \( I, J \) and \( K \) are integrable complex structures satisfying the quaternion identities, \( g \) is a hyper-Hermitian metric of signature \((4p, 4q)\) and there is an \( \text{Sp}(p, q) \)-connection \( \nabla \) whose torsion tensor

\[
c(X, Y, Z) = g(X, T(Y, Z))
\]

is totally skew, where \( T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \). The integrability of \( I \) implies

\[
T(IX, IY) = IT(IX, Y) = IT(X, IY) = T(X, Y) = 0 \tag{2.1}
\]

and that \( c \) is of type \((2, 1)_I + (1, 2)_I\). Note that for a given \((g, I, J, K)\) there is at most one HKT connection \( \nabla \), sometimes called the Bismut connection.

We set \( F_I(X, Y) = g(IX, Y) \) and define \( d_I \) on \( r \)-forms by

\[
d_I \beta = (-1)^r I dI \beta,
\]

where \( I \beta = \beta(-I, \ldots, -I \cdot) \). Similar forms and operators are defined for \( J \) and \( K \). With these conventions the torsion satisfies

\[
-c = d_I F_I = d_J F_J = d_K F_K.
\]

A potential for an HKT structure is a function such that

\[
F_I = \frac{1}{2}(dd_I + d_I d_K)\mu, \quad \text{etc.}
\]

Note that \( dd_I \mu = dI d\mu \) and \( dJ dK \mu = -JdID\mu \).

In [5, Cor. 4] it is shown that locally any hypercomplex manifold \((M, I, J, K)\) admits a compatible HKT metric with potential. On the other hand, Michelson and Strominger