Quantum Invariant for Torus Link and Modular Forms

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Abstract: We consider an asymptotic expansion of Kashaev’s invariant or of the colored Jones function for the torus link $T(2, 2m)$. We shall give $q$-series identity related to these invariants, and show that the invariant is regarded as a limit of $q$ being $N$-th root of unity of the Eichler integral of a modular form of weight $3/2$ which is related to the $su(2)_m$−2 character.

1. Introduction

Recent studies reveal an intimate connection between the quantum knot invariant and “nearly modular forms” especially with half integral weight. In Ref. 9 Lawrence and Zagier studied an asymptotic expansion of the Witten–Reshetikhin–Turaev invariant of the Poincaré homology sphere, and they showed that the invariant can be regarded as the Eichler integral of a modular form of weight $3/2$. In Ref. 19, Zagier further studied a “strange identity” related to the half-derivatives of the Dedekind $\eta$-function, and clarified a role of the Eichler integral with half-integral weight. From the viewpoint of the quantum invariant, Zagier’s $q$-series was originally connected with a generating function of an upper bound of the number of linearly independent Vassiliev invariants [17], and later it was found that Zagier’s $q$-series with $q$ being the $N$th root of unity coincides with Kashaev’s invariant [5, 6], which was shown [14] to coincide with a specific value of the colored Jones function, for the trefoil knot. This correspondence was further investigated for the torus knot, and it was shown [3] that Kashaev’s invariant for the torus knot $T(2, 2m + 1)$ also has a nearly modular property; it can be regarded as a limit $q$ being the root of unity of the Eichler integral of the Andrews–Gordon $q$-series, which is a theta series with weight $1/2$ spanning $m$-dimensional space. As the torus knot is not hyperbolic, studies of the torus knot may not be attractive for the “Volume Conjecture” [5, 14] which states that an asymptotic limit of Kashaev’s invariant coincides with the hyperbolic volume of the knot complement, but they are rather absorbing from the point of view of the number theory, $q$-series and modular forms.
Motivated by our previous result on the torus knot $T(2, 2m+1)$, we study Kashaev's invariant for the torus link $T(2, 2m)$ (see Fig. 1) in this article. We shall show that the invariant is now regarded as the half-integration or the Eichler integral of a modular form of weight $3/2$. Remarkable is that this modular form is related to the $\hat{su}(2)_{m-2}$ character. It is noted that recent studies \cite{20,21} reveal a relation with Ramanujan's mock theta functions. We also propose a $q$-series identity, which is new as far as we know, and study an asymptotic expansion thereof.

This paper is organized as follows. In Sect. 2 we construct the colored Jones polynomial for the torus link $T(2, 2m)$. Using the Jones–Wenzl idempotent, we give an explicit formula of the invariant. It is known \cite{14} that Kashaev's invariant coincides with a specific value of the colored Jones polynomial. This correspondence enables us to give an integral form of Kashaev's invariant for the torus link $T(2, 2m)$ in Sect. 3. We further give an asymptotic expansion of the invariant, and see that the invariant for $T(2, 2m)$ also has a nearly modular property. We give an explicit form of Kashaev's invariant for this torus link using the enhanced Yang–Baxter operator. Combining these results we obtain an asymptotic expansion of a certain $\omega$-series. In Sect. 4 we introduce the $q$-series related to Kashaev’s invariant for the torus link, and prove a new $q$-series identity. We study the modular property of these $q$-series, and discuss how Kashaev’s invariant for $T(2, 2m)$ may be regarded as the Eichler integral of a modular form with weight $3/2$ which is the affine $\hat{su}(2)_{m-2}$ character, in Sect. 5. In the last section, we collect some examples.

2. Colored Jones Polynomial for Torus Link $(2, 2m)$

The $N$-colored Jones polynomial for torus knot $T(m, p)$ was studied in Refs. \cite{12,15}. Following these methods, we compute the colored Jones polynomial for torus link $T(2, 2m)$ in this section. We use the Jones–Wenzl idempotent, and use following formulae (see, e.g., Ref. 10);

\[
\begin{align*}
G(1, a, b, c) &= (-1)^{(a+b-c)/2} A^{a+b-c+\frac{a^2+2ab+c^2}{2}}, \\
G(2, a, b, c) &= A^{a+b-c+\frac{a^2+2ab+c^2}{2}}, \\
G(3, a, b, c) &= (-1)^{(a+b-c)/2} A^{a+b-c+\frac{a^2+2ab+c^2}{2}}.
\end{align*}
\]