Global Regularity of Wave Maps from $\mathbb{R}^{2+1}$ to $\mathbb{H}^2$. Small Energy

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Abstract: We demonstrate that Wave Maps with smooth initial data and small energy from $\mathbb{R}^{2+1}$ to the Lobatchevsky plane stay smooth globally in time. Our method is similar to the one employed in [18]. However, the multilinear estimates required are considerably more involved and present novel technical challenges. In particular, we shall have to work with a modification of the functional analytic framework used in [30], [33], [18].

1. Formulation of The Problem and Overview

Let $(M, g)$ be a Riemannian manifold equipped with metric $g = (g_{ij})$. Also, let $\mathbb{R}^{n+1}$, $n \geq 1$, be the standard Minkowski space equipped with metric

$$(\delta_{ij}) = \text{diag}(-1, 1, \ldots, 1).$$

A classical Wave Map $u$ from $\mathbb{R}^{n+1}$ to $(M, g)$ is a smooth map which is critical with respect to the functional

$$u \mapsto \int_{\mathbb{R}^{n+1}} < \partial_\alpha u, \partial^\alpha u > g \, d\sigma.$$

The following notational conventions are used: $d\sigma$ denotes the volume measure associated with $(\delta_{ij})$, $\partial_\alpha u = u_\alpha(\partial_\alpha) \in TM$, $\alpha = 0, 1, \ldots, n$, and Einstein’s summation convention is in force. Moreover, $\partial_\theta = \delta_{\theta\rho}\partial^\rho$. In local coordinates, $u$ is seen to satisfy the following conditions:

$$\Box u^i + \Gamma^i_{jk} \partial_\alpha u^j \partial^\alpha u^k = 0,$$

where $u = (u^i)$, and $\Gamma^i_{jk}$ are the Riemann-Christoffel symbols associated with the metric $g$ and the local coordinate system. We are interested in the

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$^1$ We shall also use the convention $\partial_0 = \partial_t$. 
Cauchy problem associated with (1). Given smooth initial data \( u[0] := (u(0), \partial_t u(0)) : 0 \times \mathbb{R}^2 \to M \times TM \) at time \( t = 0 \), is there a Wave Map \( u(t, x) \) extending these globally in time?

To start with, we observe that the problem is supercritical with respect to the conserved energy provided \( n > 2 \). Thus, one expects development of singularities for ‘large initial data’. Blow-up examples are given for instance in [25], p. 102. Still, in sync with the general philosophy developed by Klainerman, e.g. in [7], one expects existence of classical Wave Maps provided the initial data are smooth and small in the critical Sobolev space \( \dot{H}^\frac{n}{2} \). Moreover, for the energy critical case \( n = 2 \), one hopes for existence of classical Wave Maps for arbitrary smooth data, provided the target \((M, g)\) is ‘sufficiently nice’. Of particular importance is the following conjecture of Klainerman, in light of its close connection to Einstein’s equations under \( U(1) \)-symmetry:

Conjecture (Klainerman). Let \((\mathbb{H}^2, dg)\) be the standard hyperbolic plane. Then classical Wave Maps originating on \( \mathbb{R}^{2+1} \) exist for arbitrary smooth initial data.

Furthermore, numerical evidence elaborated in [2] suggests development of singularities for Wave Maps from \( \mathbb{R}^{2+1} \) to \( S^2 \), provided the (smooth) data are sufficiently large, even under certain symmetry assumptions (equivariance) on the Wave Map. In this paper, we shall establish a partial result towards the conjecture stated above, namely the following small data result:

**Theorem 1.1.** Let \( \mathbb{H}^2 \) be the standard hyperbolic plane, consisting of all pairs of real numbers \( \{(x, y) \mid y > 0\} \) equipped with the metric \( dg = \frac{dx^2 + dy^2}{y^2} \). Then, given smooth initial data \( (x, y)[0] : 0 \times \mathbb{R}^2 \to \mathbb{H}^2 \) which are sufficiently small in the sense that

\[
\int_{0 \times \mathbb{R}^2} \sum_{\alpha=0}^{2} \left( \left[ \frac{\partial_\alpha x}{y} \right]^2 + \left[ \frac{\partial_\alpha y}{y} \right]^2 \right) dx < \epsilon
\]

for suitably small \( \epsilon > 0 \), there exists a classical Wave Map from \( \mathbb{R}^{2+1} \) to \( \mathbb{H}^2 \) extending these globally in time.

This result is to be seen as a further step in a long sequence of developments, whose high points are the following achievements:

1. The **subcritical case** for \( n \geq 2 \): strong local well-posedness of (1) in \( H^s \), \( s > \frac{\pi}{2} \) by Klainerman-Machedon [9] and Klainerman-Selberg [13].
2. The **critical Besov case** for \( n \geq 2 \): strong global well-posedness of (1) for initial data small in the critical Besov space \( \dot{B}_{\frac{n}{2}} \) by Tataru [33].
3. **Global regularity** for Wave Maps from \( \mathbb{R}^{n+1} \), \( n \geq 2 \) to \( S^k \), \( k \geq 1 \), provided the initial data are smooth and small in the critical Sobolev space \( \dot{H}^\frac{k}{2} \) by Tao [29, 30].

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2 To define the energy, for example isometrically embed \((M, g)\) into an ambient Euclidean space using Nash’s embedding theorem, and put

\[
\|u\|_{H^1}^2 := \sum_{\alpha=0}^{n} \int |\partial_\alpha u(t, \cdot)|^2 d\sigma,
\]

where \( |\cdot| \) denotes Euclidean length. This is easily seen to be a well-defined quantity for classical Wave Maps.

3 Einstein’s equations in vacuo under \( U(1) \)-symmetry attain the form of a Wave Map originating on a Lorentzian 2+1-manifold \( M \) to \( \mathbb{H}^2 \), coupled with an elliptic system driving the metric on \( M \). Our form of the Wave Maps problem is a highly simplified version.

4 The problem for \( n = 1 \) is globally strongly well-posed in \( H^1 \), [5]. However, it is not well-posed in the critical \( H^\frac{n}{2} \) [28].