Conjugacies for Tiling Dynamical Systems

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Received: 29 January 2004 / Accepted: 7 April 2004
Published online: 5 November 2004 – © Springer-Verlag 2004

Abstract: We consider tiling dynamical systems and topological conjugacies between them. We prove that the criterion of being of finite type is invariant under topological conjugacy. For substitution tiling systems under rather general conditions, including the Penrose and pinwheel systems, we show that substitutions are invertible and that conjugacies are generalized sliding block codes.

1. Notation and Main Results

We begin with a definition of tiling dynamical systems, in sufficient generality for this work. Let \( \mathcal{A} \) be a nonempty finite collection of compact connected sets in the Euclidean space \( \mathbb{E}^d \), sets with dense interior and boundary of zero volume. Let \( X(\mathcal{A}) \) be the set of all tilings of \( \mathbb{E}^d \) by congruent copies, which we call tiles, of the elements of the “alphabet” \( \mathcal{A} \). We assume \( X(\mathcal{A}) \) is nonempty, which is automatic for the special class of substitution tiling systems on which we will concentrate below. We label the “types” of tiles by the elements of \( \mathcal{A} \). We endow \( X(\mathcal{A}) \) with the metric

\[
m[x, y] \equiv \sup_{n \geq 1} \frac{1}{n} m_H[B_n \cap \partial x, B_n \cap \partial y],
\]

where \( B_n \) denotes the open ball of radius \( n \) centered at the origin \( O \) of \( \mathbb{E}^d \), and \( \partial x \) the union of the boundaries of all tiles in \( x \). (A ball centered at \( a \) is denoted \( B_n(a) \).) The Hausdorff metric \( m_H \) is defined as follows. Given two compact subsets \( P \) and \( Q \) of \( \mathbb{E}^d \),

\[
m_H(P, Q) = \max\{\tilde{m}(P, Q), \tilde{m}(Q, P)\},
\]

where

\[
\tilde{m}(P, Q) = \sup_{p \in P} \inf_{q \in Q} \|p - q\|,
\]

with \( \|w\| \) denoting the usual Euclidean norm of \( w \).

* Research supported in part by NSF Vigre Grant DMS-0091946
** Research supported in part by NSF Grant DMS-0071643 and Texas ARP Grant 003658-158
Under the metric \( m \) two tilings are close if they agree, up to a small Euclidean motion, on a large ball centered at the origin. The converse is also true for tiling systems with finite local complexity (as defined below): closeness implies agreement, up to small Euclidean motion, on a large ball centered at the origin [see RaS1]. Although the metric \( m \) depends on the location of the origin, the topology induced by \( m \) is Euclidean invariant. A sequence of tilings converges in the metric \( m \) if and only if its restriction to every compact subset of \( \mathbb{E}^d \) converges in \( m_H \). It is not hard to show [RW] that \( X(\mathcal{A}) \) is compact and that the natural action of the connected Euclidean group \( \mathcal{G}_E \) on \( X(\mathcal{A}) \), \((g, x) \in \mathcal{G}_E \times X(\mathcal{A}) \mapsto g[x] \in X(\mathcal{A})\), is continuous.

To include certain examples it is useful to generalize the above setup, to use what is sometimes called “colored tiles”. To make the generalization we assign a “color” from some finite set to each element of \( \mathcal{A} \), represented on each tile by a “color marking”, a line segment in the interior of the tile, of different length for different colors. We then redefine \( \partial x \) as the union of the tile boundaries and color markings in the tiling \( x \).

**Definition 1.** A tiling dynamical system is the action of \( \mathcal{G}_E \) on a closed, \( \mathcal{G}_E \)-invariant subset of \( X(\mathcal{A}) \).

We emphasize the close connection between such dynamical systems and subshifts. A subshift with \( \mathbb{Z}^d \)-action is the natural action of \( \mathbb{Z}^d \) on a compact, \( \mathbb{Z}^d \)-invariant subset \( X \) of \( \mathbb{B}^{\mathbb{Z}^d} \), for some nonempty finite set \( \mathbb{B} \). If we associate with each element of \( \mathbb{B} \) a “colored” unit cube in \( \mathbb{E}^d \), the face-to-face tilings of \( \mathbb{E}^d \) by those arrays of such cubes corresponding to the subshift \( X \) gives a tiling dynamical system which is basically the suspension of the subshift \( X \) (but with rotations of the entire tiling also permitted).

A significant difference between subshifts and tiling dynamical systems is that for (nontrivial) subshifts the group acts on a Cantor set, while the space is typically connected for interesting tiling systems. In fact, the spaces for different tiling systems need not be homeomorphic.

A major objective in dynamics is the classification of interesting subclasses up to topological conjugacy. For the class of subshifts a central theorem, due to Curtis, Lyndon and Hedlund, shows that a topological conjugacy can be represented by a sliding block code (see [LM]). For tiling dynamical systems there is a natural analogue of such a representation for which we use the same term. (Such maps are called “local” in [P] and are closely related to mutual local derivability [BSJ].)

**Definition 2.** A topological conjugacy \( \psi : X_\mathcal{A} \mapsto X_{\mathcal{A}'}, \) between tiling systems is a sliding block code if for every \( n' > 0 \) there is \( n > 0 \) such that for every \( x, y \in X_\mathcal{A} \) such that \( B_n \cap \partial x = B_n \cap \partial y \) we have \( B_{n'} \cap \partial (\psi x) = B_{n'} \cap \partial (\psi y) \).

Our first result is:

**Theorem 1.** Within the subclass of substitution tiling systems with invertible substitution, every topological conjugacy is a sliding block code.

Before defining the subclass of “substitution” tiling systems in general we present some relevant examples.

A “Penrose” tiling of the plane, Fig. 1, can be made as follows. Consider the 4 (colored) tiles of Fig. 2. Divide each tile (also called a “tile of level 0”) into 2 or 3 pieces as in Fig. 2 and rescale by a linear factor of the golden mean \( \tau = (1 + \sqrt{5})/2 \) so that each piece is the same size as the original. This yields 4 collections of tiles that we call “tiles of level 1”. Subdividing each of these tiles and rescaling gives 4 collections of tiles that we call tiles of level 2. Repeating the process \( n \) times gives tiles of level \( n \). A Penrose...