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Formal Symplectic Groupoid

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Abstract: The multiplicative structure of the trivial symplectic groupoid over $\mathbb{R}^d$ associated to the zero Poisson structure can be expressed in terms of a generating function. We address the problem of deforming such a generating function in the direction of a non-trivial Poisson structure so that the multiplication remains associative. We prove that such a deformation is unique under some reasonable conditions and we give the explicit formula for it. This formula turns out to be the semi-classical approximation of Kontsevich's deformation formula. For the case of a linear Poisson structure, the deformed generating function reduces exactly to the CBH formula of the associated Lie algebra. The methods used to prove existence are interesting in their own right as they come from an at first sight unrelated domain of mathematics: the Runge–Kutta theory of the numeric integration of ODE's.

1. Introduction

In this paper we give a formal version of the integration of Poisson manifolds by symplectic groupoids. The solution of this formal integration problem relies on the existence of a generating function for which we give here the explicit formula. This generating function turns out to be a universal Campbell–Baker–Hausdorff (CBH) formula for the non-linear case. It reduces to the usual CBH formula when the Poisson structure comes from a Lie algebra. This generating function can be interpreted as the semi-classical part of the Kontsevich deformation quantization formula. This fact reminds of us the origin of symplectic groupoids which were first introduced by Weinstein in [6], Karasev in [11], and Zakrwewski in [18] as a tool to quantize the algebra of functions on a Poisson manifold. This section is devoted to recall some basic features of the program of quantization by symplectic groupoid, to formulate the formal integration problem for Poisson

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manifolds and to state the main theorem of this article which gives a positive answer to the formal integration problem.

1.1. Quantization by symplectic groupoid. The program of quantization by symplectic groupoid is an attempt to quantize the algebra of functions on Poisson manifolds by geometric means.

It is based mainly on the belief or hope, coming from geometric quantization, that there should exist a kind of correspondence or dictionary between the world of symplectic manifold (classical level) and the world of linear spaces (quantum level). This correspondence, as explained in [1], is summarized in the following table:

Symplectic world	Linear world
M	$Q(M)$
$L \subset M$	$Q(L) \in Q(M)$
$\overline{M}$	$Q(\overline{M}) = Q(M)^*$
$Q(M \times N)$	$Q(M) \otimes Q(N)$

Here M is a symplectic manifold, $\overline{M}$ the same manifold with opposite symplectic structure, L a Lagrangian submanifold, and $Q(M)$ a complex vector space. Q stands for the "Quantization functor". In particular, canonical relations, i.e., Lagrangian submanifolds of $\overline{M} \times N$ are sent by Q to linear maps from $Q(M)$ to $Q(N)$. The main ingredient is the assumption that quantization is functorial , i.e., the composition of canonical relations should be sent to the composition of linear maps (see [16]). If such a quantization functor existed, we could ask the following question:

To what kind of symplectic manifold should we associate an algebra (i.e., a vector space with an associative product)?

Answering this question leads directly to the notion of symplectic groupoid, see [17].

Definition 1. *A symplectic groupoid is a Lie groupoid G (see [1] for a precise definition of a Lie groupoid) with a symplectic form ω for which the multiplication space $G^{(m)} = \{(x, y, x \bullet y)/x, y \in G$ are composable elements$\}$ is a Lagrangian submanifold of $\overline{G} \times \overline{G} \times G$ ($\overline{G}$ being the symplectic manifold with symplectic form $-\omega$). It can be shown (see [14]) that, given a symplectic groupoid G, there is an induced Poisson structure on the base space $G^{(0)}$. Conversely, given a Poisson manifold P we call a symplectic groupoid over P any symplectic groupoid G such that the base space $G^{(0)}$ is diffeomorphic as a Poisson manifold to P. In this case we say that G integrates P and we call integrable Poisson manifolds the Poisson manifolds for which we can find such a G.*

Applying the "Quantization functor" Q to the symplectic groupoid G, we should then get a vector space $Q(G)$ and an associative product $Q(G^{(m)})$ on it. The associativity of this product is guaranteed by the associativity of the groupoid multiplication and the functoriality of Q.

These facts suggest the following procedure to quantize Poisson manifolds P:

Step 1. Find a symplectic groupoid G such that the base $G^{(0)}$ is diffeomorphic to the Poisson manifold P .

Step 2. Quantize (geometric quantization,...) G and $G^{(m)}$ to get the quantum algebra.