White-Noise and Geometrical Optics Limits
of Wigner-Moyal Equation for Wave Beams
in Turbulent Media

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Abstract: Starting with the Wigner distribution formulation for beam wave propagation in Hölder continuous non-Gaussian random refractive index fields we show that the wave beam regime naturally leads to the white-noise scaling limit and converges to a Gaussian white-noise model which is characterized by the martingale problem associated to a stochastic differential-integral equation of the Itô type. In the simultaneous geometrical optics the convergence to the Gaussian white-noise model for the Liouville equation is also established if the ultraviolet cutoff or the Fresnel number vanishes sufficiently slowly. The advantage of the Gaussian white-noise model is that its $n$-point correlation functions are governed by closed form equations.

1. Introduction

Laser beam propagation in the turbulent atmosphere is governed by the classical wave equation with a randomly inhomogeneous refractive index field

$$n(z, x) = \bar{n}(1 + \tilde{n}(z, x)), \quad (z, x) \in \mathbb{R}^3,$$

where $\bar{n}$ is the mean and $\tilde{n}(x)$ is the fluctuation of the refractive index field. We seek the solution of the form $E(t, z, x) = \Psi(z, x) \exp [i\tilde{n}(kz - wt)] + c.c.$, where $E$ is the (scalar) electric field, $k$ and $w = kc_0/\bar{n}$ are the carrier wavenumber and frequency, respectively, with $c_0$ being the wave speed in vacuum. Here and below $z$ and $x$ denote the variables in the longitudinal and transverse directions of the wave beam, respectively.

In the paraxial approximation [24], the modulation $\Psi$ is approximated by the solution of the parabolic wave equation which after nondimensionalization with respect to some

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reference lengths $L_z$ and $L_x$ in the longitudinal and transverse directions, respectively, has this form

$$i \tilde{k} \frac{\partial \Psi}{\partial z} + \frac{\gamma}{2} \Delta \Psi + \tilde{k}^2 k_0 L_z \tilde{n}(z L_z, x L_x) \Psi = 0,$$

$$\Psi(0, x) = \Psi_0(x) \in L^2(\mathbb{R}^d), \quad d = 2$$ (1)

where $\tilde{k} = k/k_0$ is the normalized wavenumber with respect to the central wavenumber $k_0$ and $\gamma$ is the Fresnel number

$$\gamma = \frac{L_z k_0}{L_x}.$$

A widely used model for the fluctuating refractive index field $\tilde{n}$ is a spatially homogeneous random field (usually assumed to be Gaussian) with the spatial structure function

$$D_n(|\tilde{x}|) = \mathbb{E}[(\tilde{n}(\tilde{x} + \cdot) - \tilde{n}(\cdot))^2] = C_n^2 |\tilde{x}|^{2/3}, \quad |\tilde{x}| \in (\ell_0, L_0),$$

where $\ell_0$ and $L_0$ are the inner and outer scales, respectively. Here and below $\mathbb{E}$ stands for ensemble average.

The refractive index structure function has a spectral representation

$$D_n(|k|) = 8\pi \int_0^{\infty} \Phi_n(|\tilde{k}|) \left[ 1 - \frac{\sin(|k||\tilde{x}|)}{|k||\tilde{x}|} \right] |\tilde{k}|^2 d|\tilde{k}|, \quad \tilde{k} \in \mathbb{R}^{d+1}$$ (2)

with the Kolmogorov spectral density

$$\Phi_n(|\tilde{k}|) = 0.033 C_n^2 |\tilde{k}|^{-11/3}, \quad |\tilde{k}| \in (\ell_0, L_0).$$ (3)

Here the structure parameter $C_n^2$ depends in general on the temperature gradient on the scales larger than $L_0$. See, e.g., [21, 16, 5] for more sophisticated models of turbulent refractive index fields.

In this paper we will consider a general class of spectral density parametrized by $H \in (0, 1)$ and satisfying the upper bound

$$\Phi(\tilde{k}) \leq K (L_0^{-2} + |\tilde{k}|^2)^{-H-1/2-d/2} \left( 1 + \ell_0^{-2} |\tilde{k}|^2 \right)^{-2}, \quad \tilde{k} = (\xi, \mathbf{k}) \in \mathbb{R}^{d+1}, \quad d = 2$$ (4)

for some positive constant $K < \infty$. $L_0$ and $\ell_0$ in (4) are the infrared and ultraviolet cutoffs. The ultraviolet cutoff is physically due to dissipation on the small scales which normally results in a Gaussian decay factor [21]. We are particularly interested in the regime where the ratio $L_0/\ell_0$ is large as in the high Reynolds number turbulent atmosphere.

Let us introduce the non-dimensional parameters that are pertinent to our scaling:

$$\varepsilon = \sqrt{\frac{L_x}{L_z}}, \quad \eta = \frac{L_x}{L_0}, \quad \rho = \frac{L_x}{\ell_0}.$$

In terms of the parameters and the power-law spectrum in (4) we rewrite (1) as

$$i \tilde{k} \frac{\partial \Psi^\varepsilon}{\partial z} + \frac{\gamma}{2} \Delta \Psi^\varepsilon + \frac{\tilde{k}^2 \mu}{\varepsilon^2} V\left( \frac{z}{\varepsilon}, x \right) \Psi^\varepsilon = 0, \quad \Psi^\varepsilon(0, x) = \Psi_0(x)$$ (5)