Compact Scalar-flat Indefinite Kähler Surfaces with Hamiltonian $S^1$-Symmetry

Hiroyuki Kamada\textsuperscript{*} \textsuperscript{**}

Numazu College of Technology, 3600 Ooka, Numazu, Shizuoka 410-8501, Japan

Received: 1 December 2002 / Accepted: 30 March 2004
Published online: 25 November 2004 – © Springer-Verlag 2004

Abstract: The existence of scalar-flat indefinite Kähler metrics on compact complex surfaces is discussed. In particular, a compact scalar-flat indefinite Kähler surface admitting a Hamiltonian $S^1$-symmetry is proved to be biholomorphic to the product of two complex projective lines, with the help of a generalization of the Bando-Calabi-Futaki character. In fact, it is shown that none of such metrics exist on Hirzebruch surfaces of positive degree. On the other hand, by employing an analogue of LeBrun’s hyperbolic ansatz, we construct a wealth of explicit scalar-flat indefinite Kähler metrics on the product of complex projective lines, and also prove that these explicit metrics provide infinitely many different isometry classes, by examining a necessary and sufficient condition for these metrics to be isometric to each other.

1. Introduction

Since the work of Ooguri-Vafa [19], Ricci-flat indefinite Kähler metrics on complex surfaces have drawn considerable attention in mathematical physics. Recently Petean [20] studied the existence of indefinite Kähler Einstein metrics, including Ricci-flat ones, on compact complex surfaces. As another important branch of generalizations of the Ricci-flat case, we are interested in the scalar-flat case. The main theme of this paper is to study the existence of scalar-flat indefinite Kähler metrics on compact complex surfaces with certain symmetry.

Let $g$ be an indefinite Hermitian metric on a complex surface $M = (M, J)$, that is, $g$ satisfies $g(J \cdot, J \cdot) = g(\cdot, \cdot)$. Then $g$ is of type $(2,2)$ (we often call an indefinite metric of neutral signature such as $g$ simply a neutral metric). A triplet $(M, J, g)$ is called an indefinite Kähler surface if the associated fundamental form $\Omega := g(J \cdot, \cdot)$ is closed. For a given (indefinite) Kähler surface $(M, J, g)$, the fundamental form $\Omega$ and

\textsuperscript{*} Supported by JSPS–MEXT. Grant-in-Aid for Young Scientists (No. 13740053).

\textsuperscript{**} Present address: The National University Corporation, Miyagi University of Education, 149 Aramaki Aoba, Aoba-ku, Sendai 980-0845, Japan.
Theorem 1. There exist scalar-flat indefinite Kähler metrics on \((M, J, g)\), respectively. Since \(g\) is indefinite, its Kähler form \(\Omega\) is a real symplectic \((1,1)\)-form on \(M\) compatible with the nonstandard (anti-complex) orientation. In this paper, we equip an indefinite Kähler surface \((M, J, g)\) with the complex orientation given by \(J\).

Let \(\gamma \coloneqq \text{Ric}(\cdot, \cdot)\) be the Ricci form of an indefinite Kähler surface \((M, g) = (M, J, g)\). In terms of local holomorphic coordinates \((z^1, z^2)\) of \(M\), the Kähler form \(\Omega\) and the Ricci form \(\gamma\) are expressed as \(\Omega = (\sqrt{-1}/2) \sum_{\alpha, \beta=1}^2 g_{\alpha \beta} de^\alpha \wedge dz^\beta\) and \(\gamma = -\sqrt{-1} \partial \bar{\partial} \log |\det(g_{\alpha \beta})|\), respectively. Thus \(\gamma\) is also a closed real \((1,1)\)-form on \(M\). Note that, since \(J\) is parallel with respect to the Levi-Civita connection \(\nabla\) of \((M, g)\), the cohomology class \([\gamma]\) of \(\gamma\) determines the real first Chern class \(c_1(M)\) of \(M\) by \(c_1(M) = (1/2\pi)[\gamma] \in H^{1,1}(M, \mathbb{R})\). By definition, \(g\) is scalar-flat if and only if \(\gamma \wedge \Omega \equiv 0\), which implies \(c_1(M)\)-\([\Omega]\) = 0.

A typical example of a compact scalar-flat indefinite Kähler surface is the product \(\mathbb{P}^1 \times \mathbb{P}^1\), of two complex projective lines with the indefinite product metric \(g_0 = (\cdot h_{\mathbb{P}^1}) \otimes h_{\mathbb{P}^1}\), where \(h_{\mathbb{P}^1}\) denotes the standard unit round metric on \(\mathbb{P}^1\). Note that \((\mathbb{P}^1 \times \mathbb{P}^1, g_0)\) is not only scalar-flat but also conformally-flat, that is, the Weyl conformal tensor \(W\) of \(g_0\) identically vanishes. Furthermore, it is known that the conformal structure of a conformally-flat indefinite metric is unique on \(\mathbb{P}^1 \times \mathbb{P}^1\). In fact, a compact simply-connected indefinite conformally-flat four-manifold is conformally equivalent to \((\mathbb{P}^1 \times \mathbb{P}^1, g_0)\) (see Kuiper [11]). As in the Riemannian case, it follows that an indefinite Kähler surface \((M, g)\) is scalar-flat if and only if \(g\) is self-dual, that is, its Weyl conformal tensor \(W\), regarded as an endomorphism on the vector bundle of two-forms \(\Lambda^2 = \Lambda^2 T M\), satisfies \(*_g W = W\), where \(*_g\) denotes the Hodge star operator of \((M, g)\). It is known that the product \(\mathbb{P}^1 \times \mathbb{P}^1\) of two complex projective lines and the one-point blowing-up \(\mathbb{P}^2 \# \mathbb{P}^2\) of the complex projective plane \(\mathbb{P}^2\) are typical examples of compact four-manifolds that admit no self-dual Riemannian metrics. To show these non-existence results for \(\mathbb{P}^1 \times \mathbb{P}^1\) and \(\mathbb{P}^2 \# \mathbb{P}^2\), the Hirzebruch signature formula and Kuiper’s theorem are essentially used. Although the Hirzebruch signature formula is also known in the indefinite case (cf. Matsushita-Law [14]), we cannot conclude similar non-existence results at once, since the norms in the formula are indefinite in general.

Related to complex structures, we then ask whether there exist non-conformally-flat scalar-flat indefinite Kähler metrics on \(\mathbb{P}^1 \times \mathbb{P}^1\) and \(\mathbb{P}^2 \# \mathbb{P}^2\). Concerning this existence problem for \(\mathbb{P}^1 \times \mathbb{P}^1\), we can show the following.

Theorem 1. There exist scalar-flat indefinite Kähler metrics on \(\mathbb{P}^1 \times \mathbb{P}^1\), which provide infinitely many different isometry classes.

In §4, we construct explicit scalar-flat indefinite Kähler metrics on \(\mathbb{P}^1 \times \mathbb{P}^1\) by employing an indefinite analogue of LeBrun’s hyperbolic ansatz, which is a method of constructing self-dual Riemannian metrics (cf. LeBrun [12]). By construction, each metric mentioned above has a nontrivial isometric \(S^1\)-action preserving the Kähler form. Since \(\mathbb{P}^1 \times \mathbb{P}^1\) is simply-connected, there exists a moment map of the action with respect to the Kähler form. We call such an \(S^1\)-action a Hamiltonian \(S^1\)-symmetry. Comparing their Killing vector fields, we obtain a necessary and sufficient condition for these metrics to be isometric, and show that these provide infinitely many different isometry classes on \(\mathbb{P}^1 \times \mathbb{P}^1\) in §5 (cf. [5, 6]). In regard to \(S^1\)-symmetry, we can show the following result.

Theorem 2. Let \((M, g)\) be a compact indefinite Kähler surface admitting a Hamiltonian \(S^1\)-symmetry. Suppose that the Kähler class \([\Omega]\) is orthogonal to the first Chern class.