Dynamic Scaling in Miscible Viscous Fingering

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Abstract: We consider dynamic scaling in gravity driven miscible viscous fingering. We prove rigorous one-sided bounds on bulk transport and coarsening in regimes of physical interest. The analysis relies on comparison with solutions to one-dimensional conservation laws, and new scale-invariant estimates. Our bounds on the size of the mixing layer are of two kinds: a naive bound that is sharp in the absence of diffusion, and a more careful bound that accounts for diffusion as a selection criterion in the limit of vanishingly small diffusion. The naive bound is simple and robust, but does not yield the experimental speed of transport. In a reduced model derived by Wooding [20], we prove a sharp upper bound on the size of the mixing layer in accordance with his experiments. Wooding’s model also provides an example of a scalar conservation law where the entropy condition is not the physically appropriate selection criterion.

1. Introduction

We study pattern formation and mixing generated by the gravity driven instability of an interface between two fluids in a porous medium. We may distinguish three stages in the evolution of the flow: (a) an early stage governed by the linear instability, (b) an intermediate stage with scaling behavior, and (c) a late stage. The linear stability analysis is classical [2, 9, 18] and describes the evolution in stage (a) well. The late stage (c) may be quite different depending on competing physical effects such as molecular diffusion or surface tension. Saffman and Taylor’s discovery of a family of traveling wave solutions (fingers), parametrized by \( \lambda \in [0, 1] \), has led to extensive work on finger selection [18]. Much of this work has been sophisticated linear stability and singular perturbation analyses examining the role of surface tension in selecting a finger (see [1, 19] for reviews). This analysis is directly related to the asymptotic profile (stage (c)) observed experimentally by Saffman and Taylor. It also provides a formal understanding of the stability of the coherent fingers in stage (b). More precisely, it is assumed that even when there
are many competing fingers, these are locally described by the Saffman-Taylor solution, and one of these ($\lambda = 1/2$ typically) is selected by an additional physical mechanism.

In most experiments there is a broad range of active modes and in view of the instability, one may expect the evolution in stage (b) to be unpredictable. Yet experimental and numerical work shows that despite the unpredictability of fine details, certain statistics (size of the mixing layer, finger width) satisfy robust scaling laws. Little is known analytically about this fully nonlinear and physically interesting regime.

Our goal is to obtain rigorous results on dynamic scaling for the simplest nontrivial model problem. We simplify matters by considering the gravity driven transport of a dilute solute $s$ by convection and diffusion (miscible fingering). Then one may assume that the mobility is uniform, and after suitable non-dimensionalization (see [20] for a derivation) we have the system

$$\partial_t s + \mathbf{u} \cdot \nabla s = \Delta s, \quad s \in [0, 1],$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u} + \nabla p = -se_z.$$  \hspace{1cm} (3)

The domain is $x = (y, z) \in [0, L]^{n-1} \times \mathbb{R}$, $n = 2, 3$. Equation (3) is Darcy’s law: the velocity is linearly proportional to the driving force which comprises a pressure gradient and buoyancy ($-se_z$). The Peclet number, $L$, is a measure of the strength of diffusion. It is the only external parameter. We are interested in scaling behavior that is independent of $L$ and boundary effects, and in particular the behavior as $L \to \infty$. For convenience we use periodic boundary conditions in $y$. We consider initial conditions that are small perturbations of the flat unstable stratification. Figure 1 shows four snapshots of the evolution. After an initial transient, the system develops a mixing zone with an intricate network of fingers on a mesoscopic scale. The details of fingering are sensitive to initial data, but there is a remarkable statistical regularity observed in physical [20] and numerical experiments [10]:

Fig. 1. Coarsening of fingers and bulk transport