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Covariant Poisson Brackets in Geometric Field Theory

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Abstract: We establish a link between the multisymplectic and the covariant phase space approach to geometric field theory by showing how to derive the symplectic form on the latter, as introduced by Crnković-Witten and Zuckerman, from the multisymplectic form. The main result is that the Poisson bracket associated with this symplectic structure, according to the standard rules, is precisely the covariant bracket due to Peierls and DeWitt.

1. Introduction

One of the most annoying flaws of the usual canonical formalism in field theory is its lack of manifest covariance, that is, its lack of explicit Lorentz invariance (in the context of special relativity) and more generally its lack of explicit invariance under space-time coordinate transformations (in the context of general relativity). Of course, this defect is built into the theory from the very beginning, since the usual canonical formalism represents the dynamical variables of classical field theory by functions on some spacelike hypersurface (Cauchy data) and provides differential equations for their time evolution off this hypersurface: thus it presupposes a splitting of space-time into space and time, in the form of a foliation of space-time into Cauchy surfaces. As a result, canonical quantization leads to models of quantum field theory whose covariance is far from obvious and in fact constitutes a formidable problem: as a well known example, we may quote the efforts necessary to check Lorentz invariance in (perturbative) quantum electrodynamics in the Coulomb gauge.

These and similar observations have over many decades nourished attempts to develop a fully covariant formulation of the canonical formalism in classical field theory, which would hopefully serve as a starting point for alternative methods of quantization. Among the many ideas that have been proposed in this direction, two have come to occupy a special role. One of these is the "covariant functional formalism", based on the concept

of "covariant phase space" which is defined as the (infinite-dimensional) space of solutions of the equations of motion. This approach was strongly advocated in the 1980's by Crnković, Witten and Zuckerman [1–3] (see also [4]) who showed how to construct a symplectic structure on the covariant phase space of many important models of field theory (including gauge theories and general relativity), but the idea as such has a much longer history. The other has become known as the "multisymplectic formalism", based on the concept of "multiphase space" which is a (finite-dimensional) space that can be defined locally by associating to each coordinate q^i not just one conjugate momentum p_i but n conjugate momenta p_i^μ ($\mu = 1, \ldots, n$), where n is the dimension of the underlying space-time manifold. In coordinate form, this construction goes back to the classical work of De Donder and Weyl in the 1930's [5, 6], whereas a global formulation was initiated in the 1970's by a group of mathematical physicists, mainly in Poland [7–9] but also elsewhere [10–12], and definitely established in the 1990's [13, 14]; a detailed exposition, with lots of examples, can be found in the GIMmsy paper [15].

The two formalisms, although both fully covariant and directed towards the same ultimate goal, are of different nature; each of them has its own merits and drawbacks.

- The multisymplectic formalism is manifestly consistent with the basic principles of field theory, preserving full covariance, and it is mathematically rigorous because it uses well established methods from calculus on finite-dimensional manifolds. On the other hand, it does not seem to permit any obvious definition of the Poisson bracket between observables. Even the question of what mathematical objects should represent physical observables is not totally clear and has in fact been the subject of much debate in the literature. Moreover, the introduction of n conjugate momenta for each coordinate obscures the usual duality between canonically conjugate variables (such as momenta and positions), which plays a fundamental role in all known methods of quantization. A definite solution to these problems has yet to be found.
- The covariant functional formalism fits neatly into the philosophy underlying the symplectic formalism in general; in particular, it admits a natural definition of the Poisson bracket (due to Peierls [16] and further elaborated by DeWitt [17–19]) that preserves the duality between canonically conjugate variables. Its main drawback is the lack of mathematical rigor, since it is often restricted to the formal extrapolation of techniques from ordinary calculus on manifolds to the infinite-dimensional setting: transforming such formal results into mathematical theorems is a separate problem, often highly complex and difficult.

Of course, the two approaches are closely related, and this relation has been an important source of motivation in the early days of the theory [8]. Unfortunately, however, the tradition of developing them in parallel seems to have partly fallen into oblivion in recent years, during which important progress was made in other directions.

The present paper, based on the PhD thesis of the second author [21], is intended to revitalize this tradition by systematizing and further developing the link between the two approaches, thus contributing to integrate them into one common picture. It is organized into two main sections. In Sect. 2, we briefly review some salient features of the multisymplectic approach to geometric field theory, focussing on the concepts needed to make contact with the covariant functional approach. In particular, this requires a digression on jet bundles of first and second order as well as on the definition of both extended and ordinary multiphase space as the twisted affine dual of the first order jet bundle and the twisted linear dual of the linear first order jet bundle, respectively: this will enable us to give a global definition of the space of solutions of the equations of motion, both in