On a Penrose Inequality with Charge

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Abstract: We construct a time-symmetric asymptotically flat initial data set to the Einstein-Maxwell Equations which satisfies

\[ m - \frac{1}{2} \left( R + \frac{Q^2}{R} \right) < 0, \]

where \( m \) is the total mass, \( R = \sqrt{A/4\pi} \) is the area radius of the outermost horizon and \( Q \) is the total charge. This yields a counter-example to a natural extension of the Penrose Inequality for charged black holes.

1. Introduction

There has recently been much interest among geometers and mathematical relativists in inequalities bounding the total mass of initial data sets from below in terms of other geometrical quantities. The first such inequality is the Positive Mass Theorem \([12, 14]\). We rephrase the Riemannian version of this result as the following variational statement:

among all time-symmetric asymptotically flat initial data sets for the Einstein-Vacuum Equations, flat Euclidean 3-space is the unique minimizer of the total mass. Thus, the total mass satisfies \( m \geq 0 \) with equality if and only if the data set is isometric to \( \mathbb{R}^3 \) with the flat metric. See the next section for precise definitions.

A stronger result is the Riemannian version of the Penrose Inequality, which can be stated in a similar variational vein: among all time-symmetric asymptotically flat initial data sets for the Einstein-Vacuum Equations with an outermost minimal surface of area \( A \), the Schwarzschild slice is the unique minimizer of the total mass. In other words,

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When these results are phrased in this fashion, a natural question is whether similar variational characterizations of the other known stationary solutions of the Einstein Equations hold. In particular, one could ask whether among all asymptotically flat axisymmetric maximal gauge initial data sets for the Einstein-Vacuum Equations with an outermost minimal surface of area $A$ and angular momentum $J$, the Kerr slice is the unique minimizer of the mass. Such a statement would imply that:

$$m \geq \frac{1}{2} \left( R^2 + \frac{4J^2}{R^2} \right)^{1/2}$$

with equality if and only if the data is isometric to the Kerr slice. Since it is not known how to define the angular momentum of a finite surface, it is necessary to assume the axisymmetry of the data set. With that hypothesis, if $X$ is the generator of the axisymmetry, then the Komar integral:

$$J(S) = \frac{1}{8\pi} \int_S k_{ij} X^i n^j \, dA$$

gives a quantity which depends only on the homological type of $S$ and tends to the total angular momentum, as $S$ tends to the sphere at infinity.

A similar question can be asked with charge replacing angular momentum: is the Reissner-Nordström slice the unique minimizer of the mass among all asymptotically flat time-symmetric initial data sets for the Einstein-Maxwell Equations? This is equivalent to asking whether the following inequality holds for any data set:

$$m \geq \frac{1}{2} \left( R + \frac{Q^2}{R} \right),$$

where $Q$ is the total charge, with equality if and only if the data is a Reissner-Nordström slice. As above, the charge:

$$Q(S) = \frac{1}{4\pi} \int_S E_i n^i \, dA$$

depends only on the homological type of $S$.

When the horizon is connected, inequality (2) can be proved by using the Inverse Mean Curvature flow [6, 9]. Indeed, the argument in [9] relies simply on Geroch monotonicity of the Hawking mass — which still holds for the weak flow introduced by Huisken and Ilmanen in [6] — while keeping track of the scalar curvature term $R_g = 2(|E|^2 + |B|^2)$. However, when the horizon has several components the same argument yields only the following inequality:

$$m \geq \frac{1}{2} \max_i \left( R_i + \left( \min_i \sum_i \varepsilon_i Q_i \right) \frac{R_i}{R_i} \right),$$

where $R_i$ and $Q_i$ are the area radii and charges of the components of the horizon $i = 1, \ldots, N$, $\varepsilon_i = 0$ or 1, and the minimum is taken over all possible combinations.

It is the purpose of this paper to point out that (2) does not hold. We prove: