Modulation Equations: Stochastic Bifurcation in Large Domains

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Abstract: We consider the stochastic Swift-Hohenberg equation on a large domain near its change of stability. We show that, under the appropriate scaling, its solutions can be approximated by a periodic wave, which is modulated by the solutions to a stochastic Ginzburg-Landau equation. We then proceed to show that this approximation also extends to the invariant measures of these equations.

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1. Introduction

We present a rigorous approximation result of stochastic partial differential equations (SPDEs) by amplitude equations near a change of stability. In order to keep notations at a bearable level, we focus on approximating the stochastic Swift-Hohenberg equation by the stochastic Ginzburg-Landau equation, although our results apply to a larger class of stochastic PDEs or systems of SPDEs. Similar results are well-known in the deterministic case, see for instance [CE90, MSZ00]. However, there seems to be a lack of

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theory when noise is introduced into the system. In particular, the treatment of extended systems (i.e. when the spatial variable takes values in an unbounded domain) is still out of reach of current techniques.

In a series of recent articles [BMPS01, Blö03a, Blö03b, BH04], the amplitude of the dominating pattern was approximated by a stochastic ordinary differential equation (SODE). On a formal level or without the presence of noise, the derivation of these results is well-known, see for instance (4.31) or (5.11) in the comprehensive review article [CH93] and references therein. This approach shows its limitations on large domains, where the spectral gap between the dominating pattern and the rest of the equation becomes small. It is in particular not appropriate to explain a modulated pattern occurring in many physical models and experiments (see e.g. [Lyt96, LM99] or [CH93] for a review). The validity of the SODE-approximation is limited to a small neighbourhood of the stability change, which shrinks, as the size of the domain gets large.

For deterministic PDEs on unbounded domains it is well-known, see e.g. [CE90, MS95, KSM92, Sch96], that the dynamics of the slow modulations of the pattern can be described by a PDE which turns out to be of Ginzburg-Landau type.

Since the theory of translational invariant SPDEs on unbounded domains is still far from being fully developed, we adopt in the present article a somewhat intermediate approach, considering large but bounded domains in order to avoid the technical difficulties arising for SPDEs on unbounded domains. Note that the same approach has been used in [MSZ00] to study the deterministic Swift-Hohenberg equation. It does not seem possible to adapt the deterministic theory directly to the stochastic equation. One major obstacle is that the whole theory for deterministic PDE relies heavily on good a-priori bounds for the solutions of the amplitude equation in spaces of sufficiently smooth functions. Such bounds are unrealistic for our stochastic amplitude equation, since it turns out to be driven by space-time white noise. Its solutions are therefore only Hölder continuous in space and time for \( \alpha < \frac{1}{2} \). Nevertheless, the choice of large but bounded domains captures and describes all the essential features of how noise in the original equation enters the amplitude equation.

1.1. Setting and results. In this article, we concentrate on deriving the stochastic Ginzburg-Landau equation as an amplitude equation for the stochastic Swift-Hohenberg equation, though we expect that similar results hold for a much wider class of equations, see Remark 2.5. The Swift-Hohenberg equation is a celebrated toy model for the convective instability in the Rayleigh-Bénard convection. A formal derivation of the equation from the Boussinesq approximation of fluid dynamics can be found in [HS77].

In the following we consider solutions to

\[
\partial_t U = -(1 + \partial_x^2)^2 U + \varepsilon^2 vU - U^3 + \varepsilon^2 \xi_\varepsilon, \tag{SH}
\]

where \( U(x, t) \in \mathbb{R} \) satisfies periodic boundary conditions on \( D_\varepsilon = [-L/\varepsilon, L/\varepsilon] \). The noise \( \xi_\varepsilon \) is assumed to be real-valued homogeneous space-time noise. To be more precise \( \xi_\varepsilon \) is a distribution-valued centred Gaussian field such that

\[
E \xi_\varepsilon(x, s)\xi_\varepsilon(y, t) = \delta(t - s)q_\varepsilon(|x - y|). \tag{1.1}
\]

The family of correlation functions \( q_\varepsilon \) is assumed to converge in a suitable sense to a correlation function \( q \). One should think for the moment of \( q_\varepsilon \) as simply being the \( 2L/\varepsilon \)-periodic continuation of the restriction of \( q \) to \( D_\varepsilon \). We will state in Assumption 7.4 the precise assumptions on \( q \) and \( q_\varepsilon \). This will include space-time white noise and noise with bounded correlation length.