The Thermodynamic Pressure of a Dilute Fermi Gas*

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Abstract: We consider a gas of fermions with non-zero spin at temperature $T$ and chemical potential $\mu$. We show that if the range of the interparticle interaction is small compared to the mean particle distance, the thermodynamic pressure differs to leading order from the corresponding expression for non-interacting particles by a term proportional to the scattering length of the interparticle interaction. This is true for any repulsive interaction, including hard cores. The result is uniform in the temperature as long as $T$ is of the same order as the Fermi temperature, or smaller.

1. Introduction and Main Results

The physics of dilute gases at low temperature has received a lot of interest in the last couple of years, due to the recent experimental advances in studying these systems. Despite tremendous interest in the problem, rigorous results starting from first principles remain sparse, and often one has to rely on uncontrolled approximations to obtain quantitative information. This is true especially for dilute systems, where the interparticle interaction can not easily be taken into account using perturbation theory. Here, dilute refers to the case when the range of the interparticle interaction is small compared with the mean particle distance.

The first, although admittedly not the most interesting, question to ask is for the ground state energy of the system. In [7], Lieb and Yngvason devised a method for proving the relevant expression for dilute Bose gases. In this case, the energy per unit volume at density $\varrho$ is given by $4\pi a \varrho^2$, where $a$ is the $(s$-wave$)$ scattering length of the interaction potential, and $a^3 \varrho \ll 1$, i.e., the system is dilute. (Units are chosen such that $\hbar = 1$ and $2m = 1$, where $m$ denotes the mass of the particles.) The corresponding expression for a two-dimensional Bose gas was later proved in [8].

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Recently, it was possible to extend these methods and prove the corresponding result for fermions \[4]. That is, the ground state energy density of a dilute gas of spin \( q \) fermions is given by

\[
\frac{3}{5} \left( \frac{6\pi^2}{q} \right)^{2/3} q^{5/3} + 4\pi a q^2 \left( 1 - q^{-1} \right) + \text{higher order in } (a^3 q) .
\]  

(1.1)

As before, \( a \) denotes the scattering length. The factor \( (1 - q^{-1}) \) in the interaction energy results from the fact that only particles with different spin can exhibit \( s \)-wave scattering. The contribution from the interaction between particles of the same spin is of higher order in \( \rho \).

In this paper, we prove the analogue of (1.1) at positive temperature. Our main result is Theorem 1. We work in the grand canonical ensemble, and consider the pressure of the gas at given temperature \( T \) and chemical potential \( \mu \). We will show that, for dilute gases, the effect of the particle interaction results in a contribution

\[-4\pi a \rho \left( 1 - q^{-1} \right)\]

where \( \rho \) is now the average density. This result holds for any temperature, as long as \( T \) is not much bigger than the Fermi temperature (for the non-interacting gas), given by

\[T_F = \left( \frac{6\pi^2}{q} \right)^{2/3} \rho^{2/3}/(\ln 2).
\]

The rational behind this formula is the following: for dilute gases, the effect of the interaction reduces to two-particle \( s \)-wave scattering, which can take place only between particles of unequal spin. This is just like in the zero-temperature (ground state) case. The effect of the temperature on this scattering process is negligible, since for \( T \lesssim T_F \), the thermal wave length is of the same order (or greater) than the mean-particle distance. The aim of this paper is to make this intuition precise.

We will now describe the system in detail. For simplicity, we consider here only the case \( q = 2 \), i.e., the spin 1/2 case. The extension to \( q > 2 \) is straightforward. The Hilbert space under consideration is given by the fermionic Fock space for spin 1/2 particles,

\[\mathcal{F} = \mathcal{F}(\Lambda_L; \mathbb{C}^2).
\]

Here, \( \Lambda_L \) denotes a cube of side length \( L \). The Hamiltonian is the direct sum

\[H = \bigoplus_{N=0}^\infty H_N ,\]

with \( H_0 = 0 \), \( H_1 = -\Delta_L \), and

\[H_N = \sum_{i=1}^N -\Delta_i + \sum_{1 \leq i < j \leq N} v(x_i - x_j) .
\]

(1.2)

for \( N \geq 2 \). Here, \( \Delta \) denotes the Laplacian with Dirichlet boundary conditions on \( \Lambda_L \). Units are chosen such that \( \hbar = 1 \) and \( 2m = 1 \), where \( m \) denotes the mass of the particles. We note that both \( H \) and \( \mathcal{F} \) depend on \( L \), of course, but we suppress this dependence in our notation.

The pair potential \( v \) is assumed to be positive, radial, and of finite range \( R_0 \). It then has a finite and positive scattering length \( a \). The scattering length may be defined as follows: if \( \varphi \) is the unique solution of the zero-energy scattering equation

\[-\Delta \varphi + \frac{1}{4} v \varphi = 0
\]

subject to the boundary condition \( \lim_{|x| \to \infty} \varphi(x) = 1 \), then \( a \) is given by \( a = \lim_{|x| \to \infty} |x|(1 - \varphi(x)) \) (see Appendix A in \[8\] for details). Note that we do not assume \( v \) to be integrable, our results also apply to the case of a hard core. Note also that for a pure hard-core interaction, the scattering length is equal to the range.