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# Setting the Quantum Integrand of M-Theory*

**Daniel S. Freed**[1]**, Gregory W. Moore**[2]

[1] Department of Mathematics, University of Texas at Austin, 1 University Station, Austin, TX, 78712-0257, USA. E-mail: dafr@math.utexas.edu
[2] Department of Physics, Rutgers University, Piscataway, NJ 08855-0849, USA. E-mail: gmoore@physics.rutgers.edu

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**Abstract:** In anomaly-free quantum field theories the integrand in the bosonic functional integral—the exponential of the effective action after integrating out fermions—is often defined only up to a phase without an additional choice. We term this choice "setting the quantum integrand". In the low-energy approximation to M-theory the $E_8$-model for the $C$-field allows us to set the quantum integrand using geometric index theory. We derive mathematical results of independent interest about pfaffians of Dirac operators in $8k + 3$ dimensions, both on closed manifolds and manifolds with boundary. These theorems are used to set the quantum integrand of M-theory for closed manifolds and for compact manifolds with either temporal (global) or spatial (local) boundary conditions. In particular, we show that M-theory makes sense on arbitrary 11-manifolds with spatial boundary, generalizing the construction of heterotic M-theory on cylinders.

## Contents



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The low-energy approximation to M-theory is a refinement of classical 11-dimensional supergravity. It has a simple field content: a metric $g$, a 3-form gauge potential $C$, and a gravitino. The M-theory action contains rather subtle "Chern-Simons" terms which, on a topologically nontrivial manifold $Y$, raise delicate issues in the definition of the (exponentiated) action. Some aspects of the problem were resolved by Witten [W1]. The key ingredients are: a quantization law for $C$ and a background magnetic current induced by the fourth Stiefel-Whitney class of the underlying manifold; an expression for the exponentiated Chern-Simons terms using an $E_8$ gauge field and an associated Dirac operator in 12 dimensions; and finally a sign ambiguity in the gravitino partition function. In [DFM] the link to $E_8$ was used to construct a model for the $C$-field and define precisely the action, assuming that the metric $g$ is fixed. The present paper gives a complete treatment of the M-theory action as a function of both $C$ and $g$. Furthermore, we treat manifolds with boundary. The boundary may have several components and each component is interpreted either as a fixed time slice (*temporal boundary*) or a boundary in space (*spatial boundary*). We do not mix temporal and spatial boundary conditions. Our discussion of spatial boundaries in §4.3 generalizes the case $Y = X \times [0, 1]$, where $X$ is a closed 10-manifold, which was described in the work of Horava and Witten [HW1, HW2]. Our analysis here makes it clear that the anomaly cancellation is *local*. (As emphasized in [BM] the locality of anomaly cancellation in the Horava-Witten model is far from obvious.) In particular, we show that there is no topological obstruction to formulating M-theory on an 11-manifold with an arbitrary number of boundary components, provided an independent $E_8$ super-Yang-Mills multiplet is present on each component.

The analysis here is more than a cancellation of anomalies in M-theory. Already in [W1] Witten showed that there is a nontrivial Green-Schwarz mechanism canceling global anomalies on closed 11-manifolds. We go further and show that the anomaly is canceled *canonically*. This is a crucial distinction for the following reason. The absence of anomalies is a necessary condition for a quantum theory to be well-defined, but the cancellation mechanism depends on physically measurable choices. Put differently, there are undetermined phases if the configuration space of bosonic fields is not connected. As we explain quite generally in §4.1, the exponentiated effective action after integrating out fermionic fields is naturally a section of a hermitian line bundle with covariant derivative over the space of bosonic fields. The absence of anomalies means